Learning Fuzzy Dual-Modal Concepts and Constructing Concept Lattices Using Modal Operators in Formal Concept Analysis

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Formal Concept Analysis (FCA) is a mathematically founded method for extracting formal concepts from a binary relation between a set of objects and a set of properties (attributes), referred to as a crisp formal context. A formal concept is a pair $\langle objects, properties \rangle$ induced by the sufficiency operator and organized into a lattice structure. Since real-world data may involve fuzzy properties, several fuzzy extensions of FCA have been proposed. These approaches aim to extract fuzzy formal concepts using a fuzzy generalization of the sufficiency operator, which often leads to a combinatorial explosion in the number of induced concepts, limiting their practical applicability. Beyond sufficiency-based reasoning, modal FCA introduces necessity and possibility operators to extract modal concepts, yet it remains largely confined to crisp contexts. Extending such modalities to fuzzy settings raises the challenge of constructing concept lattices of manageable size while preserving computational feasibility. To address this issue, we propose a modal extension of FCA for fuzzy formal contexts based on necessity and possibility operators, referred to as $\mathcal{FM}$-operators, which induce fuzzy dual-modal concepts ($\mathcal{FDM}$-concepts). We establish a fundamental theorem characterizing the algebraic and lattice structure of these $\mathcal{FDM}$-concepts. To support practical computation, we develop two algorithms for extracting $\mathcal{FDM}$-concepts: a powerset-based algorithm and an incremental algorithm. Both algorithms have exponential worst-case complexity $\mathcal{O}(2^m \cdot n \cdot m)$, where n and m denote the numbers of objects and properties, while the incremental algorithm is output-polynomial in the number of extracted concepts. Experiments on synthetic and real-world datasets demonstrate that the proposed $\mathcal{FDM}$ framework significantly reduces the number of induced concepts compared to classical fuzzy FCA. We further illustrate its applicability to information retrieval, disjunctive attribute implications, and adaptive nonlinear control systems.

Povzetek: Članek predstavi fuzzy dualno-modalno razširitev FCA, ki z operatorjema nujnosti in možnosti iz fuzzy kontekstov izlušči manjše in računsko obvladljivejše rešetke konceptov kot klasični fuzzy FCA.

1 Introduction

Formal concept analysis [1] (FCA for short) is a theoretically founded method for extracting granules of knowledge, called formal concepts, from a binary relation, called a formal context, between a set of objects and a set of properties (attributes). Formal concepts are mathematically defined through powerset Galois derivation operators, also referred to as sufficiency derivation operators [2]. These operators establish a mapping between sets of objects, referred to as extents, and sets of shared properties, referred to as intents. The set of resulting formal concepts is structured as a concept lattice, which provides a hierarchical organization that enables the ordering and representation of knowledge. This mathematical foundation makes FCA an attractive area of research from a theoretical perspective [3, 4, 5, 6, 7, 8, 9, 10], and has found applications in several computer science fields, including data analysis and mining [11], decision making [12], knowledge discovery and representation [13, 14], information retrieval [15, 16], and machine learning [17].

FCA [1, 18] assumes a crisp binary formal context in which the object–property relation is Boolean, i.e., a property is either satisfied or not by an object. In practice, many real-world datasets arise from human judgment, observation, or measurement processes and therefore involve graded, uncertain, or imprecise information. To address this limitation, several fuzzy extensions of FCA have been proposed [3, 5, 19, 20]. In these approaches, the underlying binary relation is extended to a fuzzy relation taking values in a scale L, typically the unit interval $L = [0, 1]$. This extension leads to a generalization of the classical sufficiency operators based on a fuzzy implication, allowing the derivation of fuzzy formal concepts.

Despite their enhanced modeling capabilities, fuzzy FCA approaches suffer from a major practical limitation: the number of induced fuzzy formal concepts may, depending on the chosen scale (L) and fuzzy implication, grow extremely large or even become infinite, which severely limits the practical usability of fuzzy concept lattices. Several strategies have been proposed to address this issue. In particular, Bělohlávek et al. [21] restrict their analysis to fuzzy formal concepts generated in a crisp manner, whereas Zhang et al. [22] introduce threshold-based mechanisms to control the number of extracted concepts. Nevertheless, while effective in practice, these approaches raise questions regarding possible compromises between semantic expressiveness and structural control. An alternative direction has been proposed by Yahia and Jaoua [23], and independently by Krajči [4], through the introduction of alternative Galois derivation operators, in which either the extent is a crisp set and the intent is a fuzzy set, or conversely the extent is a fuzzy set and the intent is a crisp set. Notably, these approaches reduce the number of extracted formal concepts; nevertheless, they remain based exclusively on sufficiency reasoning.

Inspired by modal logic, Düntsch and Gediga [2] introduced modal-style operators into FCA, thus enabling reasoning beyond sufficiency through possibility and necessity modalities. Similar interpretations have been developed from a possibility-theoretic perspective [24], with recent applications to the extraction of disjunctive attribute implications [8]. While extending such modal reasoning to fuzzy formal contexts appears to be a natural direction, existing modal FCA approaches have been developed mainly for crisp contexts, and a direct extension to the fuzzy setting would likely face the previously identified scalability issue, namely that the number of induced fuzzy concepts may grow extremely large or even become infinite. To this end, this paper proposes a modal extension of FCA for fuzzy formal contexts that: (i) supports graded relational information, (ii) integrates modal reasoning through necessity and possibility operators, and (iii) enables the construction of concept lattices of reasonable size and structure. Accordingly, we introduce fuzzy modal operators, referred to as $\mathcal{FM}$-operators, defined within the fuzzy setting. We formally establish the algebraic properties of these operators and their compositions ($\mathcal{FDM}$-compositions). We then characterize the lattice structure of $\mathcal{FDM}$-concepts through a fundamental theorem and, in particular, propose an incremental algorithm for their efficient computation. Finally, we illustrate the relevance of the proposed approach through applications to three domains: information retrieval, adaptive and nonlinear control systems, and disjunctive attribute implications.

The paper is structured as follows. Section 2 recalls the necessary background on FCA. Section 3 introduces the proposed $\mathcal{FM}$-operators and investigates their algebraic properties. In Section 4, the lattice structure of $\mathcal{FDM}$-concepts is established through a fundamental theorem, and algorithms for generating the corresponding lattice are proposed. In addition, we evaluate the proposed approach against classical fuzzy FCA and (crisp) modal FCA. Section 5 presents three concrete application domains of the proposed framework. Finally, we conclude and outline directions for future research.

2 Preliminaries

This section recalls the basic notions of FCA required for the subsequent developments. We review the classical crisp framework and its modal extensions, followed by its main fuzzy generalizations.

2.1 Crisp settings

A crisp formal context is a triple $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, where $\mathcal{O} = \{o_1, o_2, \ldots, o_n\}$ and $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ are finite sets of objects and properties (or attributes) , respectively, and $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{P}$ is a crisp binary relation. Formally, $\mathcal{R}$ is defined as a mapping $\mathcal{R} : \mathcal{O} \times \mathcal{P} \to \{0, 1\}$, where $\mathcal{R}(o, p) = 1$ (resp. $\mathcal{R}(o, p) = 0$) indicates that the object o satisfies (resp. does not satisfy) the property p.

In the original proposal [1, 18], a formal concept is defined as a pair $\langle O, P\rangle$, where O, called the extent, is a subset of objects, and P, called the intent, is a subset of properties (or attributes). The intent consists exactly of those properties shared by all objects in the extent, while the extent contains exactly those objects that satisfy all properties in the intent. Formally, this corresponds to the conditions $O^{\uparrow} = P$ and $P^{\downarrow} = O$, where the derivation operators $(\cdot)^{\uparrow} : 2^{\mathcal{O}} \to 2^{\mathcal{P}}$ and $(\cdot)^{\downarrow} : 2^{\mathcal{P}} \to 2^{\mathcal{O}}$ are defined as follows:

$$O^{\uparrow} = \{p \in \mathcal{P} \mid \forall o \in \mathcal{O}, o \in O \Rightarrow \mathcal{R}(o, p) = 1\} \quad (1)$$

$$P^{\downarrow} = \{o \in \mathcal{O} \mid \forall p \in \mathcal{P}, p \in P \Rightarrow \mathcal{R}(o, p) = 1\} \quad (2)$$

Throughout the remainder of this paper, the operators $(.)^{\uparrow}$ and $(.)^{\downarrow}$ are referred to as sufficiency operators [2].

Given a formal context $\mathcal{K}$, the set of all formal concepts of $\mathcal{K}$, equipped with the partial order $\leq$ defined by $\langle O_1, P_1\rangle \leq \langle O_2, P_2\rangle$ iff $O_1 \subseteq O_2$ (equivalently, $P_2 \subseteq P_1$), forms a complete lattice, called the concept lattice of $\mathcal{K}$ and denoted by $\mathfrak{L}(\mathcal{K})$. The algebraic structure of $\mathfrak{L}(\mathcal{K})$ is characterized by the fundamental theorem of FCA [18].

Several algorithms (e.g., *Next-Closure* [25], *Close-by-One* [26], and *In-Close* [27]) have been proposed to compute the set of all formal concepts of $\mathcal{K}$, whose cardinality may reach $2^{\min(|\mathcal{O}|, |\mathcal{P}|)}$ in the worst case. These algorithms generally operate with output-polynomial complexity while relying exclusively on the classical sufficiency operators and thus do not support modalities such as necessity and possibility.

Inspired by modal logic, Düntsch and Gediga [2] introduced a pair of dual modal-style operators into classical FCA, namely a possibility operator (denoted $\langle\mathcal{R}\rangle$) and its dual necessity operator (denoted $[\mathcal{R}]$). For any subset of objects $O \in 2^{\mathcal{O}}$, these operators are defined as:

$$\langle\mathcal{R}\rangle(O) = \{p \in \mathcal{P} \mid O \cap R(p) \neq \emptyset\} \quad (3)$$

$$[\mathcal{R}]\,(O) = \{p \in \mathcal{P} \mid R(p) \subseteq O\} \quad (4)$$

$\langle\mathcal{R}\rangle(O)$ is the set of properties satisfied by at least one object in O. $[\mathcal{R}](O)$ is the set of properties such that any object satisfying one of them is necessarily in O.

2.2 Fuzzy settings

The fuzzy extension of FCA relies on fuzzy set theory, where object–property relations are represented by membership degrees. We briefly recall below the notion of fuzzy sets.

Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ be a universe of discourse and let $L^{\mathcal{U}}$ (typically $L = [0, 1]$) denote the set of all fuzzy subsets of $\mathcal{U}$. A fuzzy set $\widetilde{F} \in L^{\mathcal{U}}$ is defined by a membership function $\mu_{\widetilde{F}} : \mathcal{U} \to L$, where $\mu_{\widetilde{F}}(u)$ denotes the membership degree of $u \in \mathcal{U}$ in $\widetilde{F}$. In this paper, we represent $\widetilde{F}$ as $\widetilde{F} = \{u_1^{\alpha_1}, u_2^{\alpha_2}, \ldots, u_n^{\alpha_n}\}$, where $\alpha_i = \mu_{\widetilde{F}}(u_i)$. A fuzzy formal context (also called an L-context) is a tuple $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ where the fuzzy relation $\mathcal{I}$ is a mapping $\mathcal{I} : \mathcal{O} \times \mathcal{P} \to L$, and $\mathcal{I}(o, p)$ is interpreted as the (gradual) degree to which the object o satisfies the property p. The fuzzy generalization of the sufficiency operators $(.)^{\uparrow}$ and $(.)^{\downarrow}$ is based on fuzzy implications [3] and is expressed as mappings between $L^{\mathcal{O}}$ and $L^{\mathcal{P}}$ as follows:

$$\widetilde{O}^{\uparrow}(p) = \bigwedge_{o\in\mathcal{O}} \left(\widetilde{O}(o) \hookrightarrow \mathcal{I}(o,p)\right) \quad (5)$$

$$\widetilde{P}^{\downarrow}(o) = \bigwedge_{p\in\mathcal{P}} \left(\widetilde{P}(p) \hookrightarrow \mathcal{I}(o,p)\right) \quad (6)$$

A fuzzy formal concept $\langle\widetilde{O}, \widetilde{P}\rangle$ consists of a pair of fuzzy set of objects $\widetilde{O}$ and fuzzy set of properties $\widetilde{P}$ such that $\widetilde{O}^{\uparrow} = \widetilde{P}$ and $\widetilde{P}^{\downarrow} = \widetilde{O}$. A further interesting feature of both sufficiency operators $(.)^{\uparrow}$ and $(.)^{\downarrow}$ is that a closure property [28] of the compositions $((.)^{\uparrow})^{\downarrow}$ and $((.)^{\downarrow})^{\uparrow}$ is implicitly conveyed under some algebraic requirements. Thus, the development of a fuzzy FCA theory requires an appropriate fuzzy truth structure, or fuzzy algebra [29]. Such structure is endowed by logical connectives and represented by $(L, \wedge, \vee, 1, 0, \hookrightarrow)$ [3]. It has also been established in [3] that, under appropriate algebraic requirements, the set of all fuzzy formal concepts forms a complete lattice (denoted $\mathfrak{L}(\widetilde{\mathcal{K}})$).

Example 1. *Let us consider the L-context* $\widetilde{\mathcal{K}} = (L, \{Paul, Sandy, Mike, Nelly\}, \{Python, Java, C\#\}, \mathcal{I})$. *The fuzzy relation* $\mathcal{I}$*, given in Table 1, indicates to which extent a given student masters a given programming language.*
For instance, let us also consider a fuzzy algebra $(L, \wedge, \vee, 1, 0, \hookrightarrow)$ *such that* $L = [0, 0.1, 0.2, \ldots, 0.9, 1]$ *is a finite scale, and* $\hookrightarrow$ *is chosen as the* Łukasiewicz *fuzzy implication (i.e.* $p \hookrightarrow q = Min(1 - p + q, 1)$, $\forall p, q \in [0, 1]$). *It follows that the pair* $\langle\{Paul^{0.5}, Sandy^{0.8}, Mike^{0.3}, Nelly^{0.4}\}, \{Python^{1.0}, Java^{0.9}, C\#^{0.7}\}\rangle$ *is a fuzzy formal concept .*

The number of induced fuzzy formal concepts strongly depends on the choice of both the fuzzy implication and

Table 1: Example of L-context

$\mathcal{I}$	$Python$	$Java$	$C\#$
$Paul$	0.9	0.7	0.2
$Sandy$	0.8	0.7	0.5
$Mike$	0.3	1.0	0.8
$Nelly$	0.4	0.6	0.1

the scale L. Consequently, this number may become extremely large. In such a case, the lattice $\mathfrak{L}(\widetilde{\mathcal{K}})$ becomes impractical for concrete applications. For instance, Table 2 reports the number of fuzzy formal concepts induced by the L-context of Example 1, depending on the choice of three finite scales and two fuzzy implications. We assume $L_1 = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$, $L_2 = \{0, 0.01, 0.02, \ldots, 0.99, 1\}$, and $L_3 = \{0, 0.001, 0.002, \ldots, 0.999, 1\}$ together with the *Łukasiewicz* and *Goguen* fuzzy implications. For the computation of fuzzy formal concepts, classical FCA algorithms such as *Next-Closure* and *In-Close* have been extended to the fuzzy setting, yielding variants such as *Fuzzy Next-Closure* [30, 31] and *Fuzzy In-Close* [32]. Although these algorithms typically operate with output-polynomial complexity, the number of induced fuzzy formal concepts may reach $|L|^{\min(|\mathcal{O}|,|\mathcal{P}|)}$ in the worst case, which may lead to a combinatorial explosion and make the complete computation impractical for large fuzzy contexts, even when the scale L is finite. This intrinsic computational complexity motivates the use of restriction or reduction strategies to limit the number of induced fuzzy formal concepts [22, 31]. In particular, Yahia and Jaoua [23] and, independently, Krajči [4] proposed mappings that yield one-sided fuzzy concepts, where either the extent is crisp and the intent is fuzzy ($\langle O, \widetilde{P}\rangle$), or the extent is fuzzy and the intent is crisp ($\langle\widetilde{O}, P\rangle$), leading to a significant reduction in the number of induced concepts. For instance, the pair of mappings proposed in [23], namely $f : 2^{\mathcal{O}} \to L^{\mathcal{P}}$ and $h : L^{\mathcal{P}} \to 2^{\mathcal{O}}$, is defined as follows:

$$f(O)(p) = \bigwedge_{o\in O} \mathcal{I}(o,p), \quad (7)$$

$$h(\widetilde{P}) = \{o \in \mathcal{O} \mid \forall p \in \mathcal{P}, \widetilde{P}(p) \leq \mathcal{I}(o,p)\}. \quad (8)$$

The pair (f, h), as illustrated in Example 2, drastically reduces the number of induced concepts while preserving a complete lattice structure [4], thus making this approach more suitable for practical applications.

Example 2. *From the L-context given in Example 1, the pair of mappings* $(f,\ h)$ *allows us to extract a reduced number (i.e. 9) of crisp-fuzzy formal concepts, among them, the pair* $\langle\{Paul, Sandy\ Nelly\}, \{Python^{0.4}, Java^{0.6}, C\#^{0.1}\}\rangle$.

2.3 Scalability issues in modal-fuzzy FCA

The fuzzy extension of FCA enhances the crisp approach by allowing graded object–attribute relations over a scale L.

Table 2: The number of fuzzy formal concepts induced by the L-context of the Example 1.

Fuzzy implication ($x \hookrightarrow y$)	L_1	L_2	L_3
Łukasiewicz($= Min(1-x+y,1)$)	133	63241	57611401
Goguen ($= 1$ if $x \leq y; x/y$ elsewhere)	187	93199	85968181

Such an extension may lead to increased combinatorial complexity and uncontrolled lattice growth, as the number of induced concepts depends on the size of the fuzzy context, the cardinality of L, and the chosen fuzzy implication. Given these observations, Table 3 provides a comparative overview of crisp, fuzzy, and modal FCA approaches with respect to induced concept growth, computational complexity, and the support of fuzziness and modalities.

Extending modal FCA to a fuzzy setting constitutes a natural and promising direction. Nevertheless, existing approaches have been proposed exclusively for crisp contexts [2, 8, 24]. Their direct generalization would likely lead to scalability issues comparable to those arising in fuzzy FCA, as the number of induced fuzzy concepts may become excessively large or even infinite. Therefore, scalable approaches are required to ensure controlled lattice size and computational feasibility in fuzzy modal FCA. Accordingly, we propose a fuzzy modal framework based on appropriate fuzzy modal operators, referred to as $\mathcal{FM}$-operators, designed to induce a lattice of fuzzy dual-modal concepts (denoted $\mathcal{FDM}$-concepts) with controlled size and structure.

3 $\mathcal{FM}$-operators

This section introduces fuzzy modal operators ($\mathcal{FM}$-operators) and establishes the algebraic properties required to induce lattices of fuzzy dual-modal concepts.

3.1 Notation and basic definitions

We first fix the notation and present the basic definitions of $\mathcal{FM}$-operators. Table 4 summarizes the main symbols used in the paper.

Depending on whether the modal necessity or possibility operator is applied to fuzzy or crisp sets of objects or properties, eight $\mathcal{FM}$-operators can be defined. In this paper, we consider the case of fuzzy sets of objects and crisp sets of properties. The superscripts determine the modal interpretation of the operators: $\square$ and $\blacksquare$ correspond to possibility, whereas $\Diamond$ and $\blacklozenge$ correspond to necessity. Accordingly, four $\mathcal{FM}$-operators are introduced in the following definition.

Definition 1. ($\mathcal{FM}$ – operators) *Let $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ be an L-context. The $\mathcal{FM}$-operators $(.)^{\uparrow^{\Diamond}} : L^{\mathcal{O}} \to 2^{\mathcal{P}}$, $(.)^{\downarrow^{\blacksquare}} : 2^{\mathcal{P}} \to L^{\mathcal{O}}$, $(.)^{\uparrow^{\square}} : L^{\mathcal{O}} \to 2^{\mathcal{P}}$, and $(.)^{\downarrow^{\blacklozenge}} : 2^{\mathcal{P}} \to L^{\mathcal{O}}$ are defined as:*

$$\widetilde{O}^{\uparrow^{\Diamond}} = \{p \in \mathcal{P} | \forall o \in \mathcal{O}, \mathcal{I}(o,p) \geq 1 - \widetilde{O}(o)\} \quad (9)$$

$$P^{\downarrow^{\blacksquare}}(o) = \bigwedge_{p \in \mathcal{P} \setminus P} (1 - \mathcal{I}(o,p)) \quad (10)$$

$$\widetilde{O}^{\uparrow^{\square}} = \{p \in \mathcal{P} \mid \forall o \in \mathcal{O}, \mathcal{I}(o,p) \leq \widetilde{O}(o)\} \quad (11)$$

$$P^{\downarrow^{\blacklozenge}}(o) = \bigvee_{p \in P} \mathcal{I}(o,p) \quad (12)$$

3.2 Algebraic properties

In the remainder of this paper, we shall mainly be interested by algebraic properties of both $(.)^{\uparrow^{\square}}$ and $(.)^{\downarrow^{\blacklozenge}}$ $\mathcal{FM}$-operators. The remaining operators will be the subject to further research. The Proposition 1 establishes that both $(.)^{\uparrow^{\square}}$ and $(.)^{\downarrow^{\blacklozenge}}$ are inclusion-preserving operators. We recall that a mapping $\Phi : \mathcal{U} \to \mathcal{V}$ is said to be inclusion-preserving if the implication $\forall A, B \in \mathcal{U}, A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B)$ holds.

Proposition 1.

(P1) : $(.)^{\uparrow^{\square}}$ *is an inclusion-preserving operator.*

(P2) : $(.)^{\downarrow^{\blacklozenge}}$ *is an inclusion-preserving operator.*

Proof.
(P1): Given $\widetilde{O}_1, \widetilde{O}_2 \in L^{\mathcal{O}}$, we have to prove that : $\widetilde{O}_1 \subseteq \widetilde{O}_2 \Longrightarrow \widetilde{O}_1^{\uparrow^{\square}} \subseteq \widetilde{O}_2^{\uparrow^{\square}}$
We have: $\widetilde{O}_1^{\uparrow^{\square}} = \{p \in \mathcal{P} \mid \forall o \in \mathcal{O}, \mathcal{I}(o,p) \leq \widetilde{O}_1(o)\}$ and $\widetilde{O}_2^{\uparrow^{\square}} = \{p \in \mathcal{P} \mid \forall o \in \mathcal{O}, \mathcal{I}(o,p) \leq \widetilde{O}_2(o)\}$, thus,

$$\begin{aligned}
\widetilde{O}_1 \subseteq \widetilde{O}_2 \quad &\Longrightarrow \quad \forall o \in \mathcal{O}, \widetilde{O}_1(o) \leq \widetilde{O}_2(o) \\
&\Longrightarrow \quad \forall p \in \widetilde{O}_1^{\uparrow^{\square}}, \forall o \in \mathcal{O}, \\
&\qquad\qquad \mathcal{I}(o,p) \leq \widetilde{O}_1(o) \leq \widetilde{O}_2(o) \\
&\Longrightarrow \quad \forall p \in \mathcal{P}, p \in \widetilde{O}_1^{\uparrow^{\square}} \Rightarrow p \in \widetilde{O}_2^{\uparrow^{\square}} \\
&\Longrightarrow \quad \widetilde{O}_1^{\uparrow^{\square}} \subseteq \widetilde{O}_2^{\uparrow^{\square}}.
\end{aligned}$$

(P2): Given $P_1, P_2 \in 2^{\mathcal{P}}$, we have to prove that : $P_1 \subseteq P_2 \Rightarrow (P_1)^{\downarrow^{\blacklozenge}} \subseteq (P_2)^{\downarrow^{\blacklozenge}}$.
First, recall that in fuzzy set theory the ordering on fuzzy sets is defined pointwise: for all $\widetilde{A}, \widetilde{B} \in L^{\mathcal{O}}$, $\widetilde{A} \subseteq \widetilde{B}$ iff $\widetilde{A}(o) \leq \widetilde{B}(o)$ for all $o \in \mathcal{O}$.
Suppose $P_1 \subseteq P_2$, and let $o \in \mathcal{O}$ be an arbitrary object. We define :

$$\alpha_1 = \bigvee_{p \in P_1} \mathcal{I}(o,p) \quad \text{and} \quad \alpha_2 = \bigvee_{p \in P_2 \setminus P_1} \mathcal{I}(o,p)$$

Depending on the values of α_1 and α_2, both situations are possible:

Table 3: Comparative summary of FCA approaches and their structural limitations

Approach	Data	Reasoning	Number of Induced Concepts	Computational Complexity
Crisp FCA	Binary	Sufficiency	Up to $2^{\min(\vert\mathcal{O}\vert,\vert\mathcal{P}\vert)}$	Output-polynomial
Fuzzy FCA	Fuzzy	Sufficiency	Up to $\vert L\vert^{\min(\vert\mathcal{O}\vert,\vert\mathcal{P}\vert)}$	Output-polynomial
Modal FCA	Binary	Modal	Up to $2^{\min(\vert\mathcal{O}\vert,\vert\mathcal{P}\vert)}$	Output-polynomial

Table 4: Summary of notation

Notation	Description
$L = [0, 1]$	Unit interval
$\mathcal{O}$	The set of objects
$\mathcal{P}$	The set of properties (or attributes)
$\mathcal{I} : \mathcal{O} \times \mathcal{P} \to L$	Fuzzy object–property relation
$\mathcal{I}(o, p)$	Degree of satisfaction of property p by object o
$2^{\mathcal{X}}$	Power set of $\mathcal{X}$; $\mathcal{X} \in \{\mathcal{O}, \mathcal{P}\}$
$X \in 2^{\mathcal{X}}$	Crisp subset of $\mathcal{X}$
$L^{\mathcal{X}}$	Set of L-valued subsets of $\mathcal{X}$
$\widetilde{X} \in L^{\mathcal{X}}$	$\widetilde{X}$ is a fuzzy subset of $\mathcal{X}$
$\widetilde{X}(x)$	Membership degree of x in $\widetilde{X}$

– If $\alpha_1 \geq \alpha_2$, then the addition of attributes from $P_2 \setminus P_1$ does not increase the degree for o, and the result remains α_1. That is, $(P_1)^{\downarrow^{\blacklozenge}}(o) = (P_2)^{\downarrow^{\blacklozenge}}(o)$

– If $\alpha_1 < \alpha_2$, then the extended set P_2 increases the supremum, and the resulting value is α_2. That is, $(P_1)^{\downarrow^{\blacklozenge}}(o) < (P_2)^{\downarrow^{\blacklozenge}}(o)$

This shows that adding attributes to P_1 does not decrease the fuzzy degree assigned to an object under the supremum aggregation, which supports the inclusion-preserving property of the operator $(.)^{\downarrow^{\blacklozenge}}$. □

The following proposition addresses the problem of decomposition w.r.t. intersection and union for both $(.)^{\uparrow^{\Box}}$ and $(.)^{\downarrow^{\blacklozenge}}$ $\mathcal{FM}$-operators.

Proposition 2. *Let $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ be an L-context, the following properties hold $\forall\, \widetilde{O}_1, \widetilde{O}_2 \in L^{\mathcal{O}}$, $\forall\, P_1, P_2 \in 2^{\mathcal{P}}$:*

$$\begin{aligned}
&(\text{P3}) : (\widetilde{O}_1 \bigvee \widetilde{O}_2)^{\uparrow^{\Box}} \supseteq \widetilde{O}_1^{\uparrow^{\Box}} \bigcup \widetilde{O}_2^{\uparrow^{\Box}}\\
&(\text{P3}') : (P_1 \bigcup P_2)^{\downarrow^{\blacklozenge}} = P_1^{\downarrow^{\blacklozenge}} \bigvee P_2^{\downarrow^{\blacklozenge}}\\
&(\text{P4}) : (\widetilde{O}_1 \bigwedge \widetilde{O}_2)^{\uparrow^{\Box}} = \widetilde{O}_1^{\uparrow^{\Box}} \bigcap \widetilde{O}_2^{\uparrow^{\Box}}\\
&(\text{P4}') : (P_1 \bigcap P_2)^{\downarrow^{\blacklozenge}} \subseteq P_1^{\downarrow^{\blacklozenge}} \bigwedge P_2^{\downarrow^{\blacklozenge}}
\end{aligned}$$

Proof.

(P3):

$$\begin{aligned}
\widetilde{O}_1^{\uparrow^{\Box}} \bigcup \widetilde{O}_2^{\uparrow^{\Box}} &= \{p \in \mathcal{P} \mid \forall o \in \mathcal{O},\, \mathcal{I}(o,p) \leq \widetilde{O}_1(o)\}\\
&\quad \bigcup \{p \in \mathcal{P} | \forall o \in \mathcal{O},\, \mathcal{I}(o,p) \leq \widetilde{O}_2(o)\}\\
&\subseteq \{p \in \mathcal{P} \mid \forall o \in \mathcal{O},\, \mathcal{I}(o,p) \leq \widetilde{O}_1(o)\\
&\quad \textit{or } \mathcal{I}(o,p) \leq \widetilde{O}_2(o)\}\\
&\subseteq \{p \in \mathcal{P} \mid \forall o \in \mathcal{O},\\
&\quad \mathcal{I}(o,p) \leq \widetilde{O}_1(o) \bigvee \widetilde{O}_2(o)\}\\
&\subseteq (\widetilde{O}_1 \bigvee \widetilde{O}_2)^{\uparrow^{\Box}}.
\end{aligned}$$

$(\text{P3}')$: we have to prove that:

$$\forall\, o \in \mathcal{O},\, (P_1 \bigcup P_2)^{\downarrow^{\blacklozenge}}(o) = P_1^{\downarrow^{\blacklozenge}}(o) \bigvee P_2^{\downarrow^{\blacklozenge}}(o)$$

$$\begin{aligned}
(P_1 \bigcup P_2)^{\downarrow^{\blacklozenge}}(o) &= \bigvee_{p \in P_1 \cup P_2} \mathcal{I}(o,p)\\
&= \bigvee \{ \bigvee_{p \in P_1} \mathcal{I}(o,p),\, \bigvee_{p \in P_2} \mathcal{I}(o,p)\}\\
&= P_1^{\downarrow^{\blacklozenge}}(o) \bigvee P_2^{\downarrow^{\blacklozenge}}(o).
\end{aligned}$$

(P4) :

$$\begin{aligned}
\widetilde{O}_1^{\uparrow^{\Box}} \bigcap \widetilde{O}_2^{\uparrow^{\Box}} &= \{p \in \mathcal{P} | \forall o \in \mathcal{O},\, \mathcal{I}(o,p) \leq \widetilde{O}_1(o)\}\\
&\quad \bigcap \{p \in \mathcal{P} | \forall o \in \mathcal{O},\, \mathcal{I}(o,p) \leq \widetilde{O}_2(o)\}\\
&= \{p \in \mathcal{P} \mid \forall o \in \mathcal{O},\, \mathcal{I}(o,p) \leq \widetilde{O}_1(o)\\
&\quad \text{and } \mathcal{I}(o,p) \leq \widetilde{O}_2(o)\}\\
&= \{p \in \mathcal{P} \mid \forall o \in \mathcal{O},\\
&\quad \mathcal{I}(o,p) \leq \widetilde{O}_1(o) \bigwedge \widetilde{O}_2(o)\}\\
&= (\widetilde{O}_1 \bigwedge \widetilde{O}_2)^{\uparrow^{\Box}}.
\end{aligned}$$

$(\text{P4}')$: we have to prove that:

$$\forall\, o \in \mathcal{O},\, (P_1 \bigcap P_2)^{\downarrow^{\blacklozenge}}(o) \leq P_1^{\downarrow^{\blacklozenge}}(o) \bigwedge P_2^{\downarrow^{\blacklozenge}}(o)$$

$$\begin{aligned}
&\forall\, o \in \mathcal{O}, \forall\, p \in P_1 \cap P_2 : \\
&\quad \begin{cases} \bigvee_{\acute{p}\in P_1} \mathcal{I}(o,\acute{p}) \geq \mathcal{I}(o,p), & \text{since } p \in P_1 \\ \bigvee_{\acute{p}\in P_2} \mathcal{I}(o,\acute{p}) \geq \mathcal{I}(o,p), & \text{since } p \in P_2 \end{cases} \\
&\quad \Longrightarrow \Big(\bigvee_{\acute{p}\in P_1} \mathcal{I}(o,\acute{p})\Big) \bigwedge \Big(\bigvee_{\acute{p}\in P_2} \mathcal{I}(o,\acute{p})\Big) \geq \mathcal{I}(o,p) \\
&\quad \Longrightarrow P_1^{\downarrow\blacklozenge}(o) \bigwedge P_2^{\downarrow\blacklozenge}(o) \geq \mathcal{I}(o,p) \\
&\quad \Longrightarrow P_1^{\downarrow\blacklozenge}(o) \bigwedge P_2^{\downarrow\blacklozenge}(o) \geq \bigvee_{\acute{p}\in P_1 \bigcap P_2} \mathcal{I}(o,\acute{p}) \\
&\quad \Longrightarrow P_1^{\downarrow\blacklozenge}(o) \bigwedge P_2^{\downarrow\blacklozenge}(o) \geq (P_1 \cap P_2)^{\downarrow\blacklozenge}(o).
\end{aligned}$$

□

3.3 $\mathcal{FDM}$-compositions

Based on the previously introduced $\mathcal{FM}$-operators, we consider the fuzzy dual-modal compositions ($\mathcal{FDM}$-compositions) between $^{\uparrow\square}$ and $^{\downarrow\blacklozenge}$. In particular, we study the compositions $((.)^{\uparrow\square})^{\downarrow\blacklozenge}$ and $((.)^{\downarrow\blacklozenge})^{\uparrow\square}$ that convey relevant properties, as given in the following proposition:

Proposition 3. *Let $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ be an L-context, and let $\widetilde{O}, \widetilde{O}_1, \widetilde{O}_2 \in L^{\mathcal{O}}$, $P, P_1, P_2 \in 2^{\mathcal{P}}$. The following properties hold:*

$$\begin{aligned}
&(\text{P5}) : \widetilde{O}_1 \subseteq \widetilde{O}_2 \Longrightarrow (\widetilde{O}_1^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq (\widetilde{O}_2^{\uparrow\square})^{\downarrow\blacklozenge} \\
&(\text{P5}') : P_1 \subseteq P_2 \Longrightarrow (P_1^{\downarrow\blacklozenge})^{\uparrow\square} \subseteq (P_2^{\downarrow\blacklozenge})^{\uparrow\square} \\
&(\text{P6}) : (\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq \widetilde{O} \\
&(\text{P6}') : P \subseteq (P^{\downarrow\blacklozenge})^{\uparrow\square} \\
&(\text{P7}) : (((\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge} = (\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge} \\
&(\text{P7}') : (((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square} = (P^{\downarrow\blacklozenge})^{\uparrow\square}
\end{aligned}$$

Proof.
(P5): Applying successively (P1) and (P2), we get :

$$\widetilde{O}_1 \subseteq \widetilde{O}_2 \Longrightarrow \widetilde{O}_1^{\uparrow\square} \subseteq \widetilde{O}_2^{\uparrow\square} \Longrightarrow (\widetilde{O}_1^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq (\widetilde{O}_2^{\uparrow\square})^{\downarrow\blacklozenge}.$$

(P5$'$) : Applying successively (P2) and (P1), we get:

$$P_1 \subseteq P_2 \Longrightarrow P_1^{\downarrow\blacklozenge} \subseteq P_2^{\downarrow\blacklozenge} \Longrightarrow (P_1^{\downarrow\blacklozenge})^{\uparrow\square} \subseteq (P_2^{\downarrow\blacklozenge})^{\uparrow\square}.$$

(P6) :

$$\begin{aligned}
&\forall \widetilde{O} \in L^{\mathcal{O}} \widetilde{O}^{\uparrow\square} = \{p \in \mathcal{P} \mid \forall o \in \mathcal{O}, \mathcal{I}(o,p) \leq \widetilde{O}(o)\} \\
&\Longrightarrow \forall o \in \mathcal{O}, \forall p \in \widetilde{O}^{\uparrow\square}, \mathcal{I}(o,p) \leq \widetilde{O}(o) \\
&\Longrightarrow \forall o \in \mathcal{O}, (\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge}(o) = \bigvee_{p\in\widetilde{O}^{\uparrow\square}} \mathcal{I}(o,p) \leq \widetilde{O}(o) \\
&\Longrightarrow \forall o \in \mathcal{O}, (\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge}(o) \leq \widetilde{O}(o) \\
&\Longleftrightarrow (\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq \widetilde{O}.
\end{aligned}$$

(P6$'$) :

$$\begin{aligned}
&P^{\downarrow\blacklozenge} = \{o^{\alpha} \mid \alpha = \bigvee_{p\in P} \mathcal{I}(o,p)\} \\
&\Longrightarrow \forall o \in \mathcal{O}, \forall p \in P, \mathcal{I}(o,p) \leq P^{\downarrow\blacklozenge}(o) \\
&\Longrightarrow \forall p \in \mathcal{P}, p \in P \Rightarrow p \in (P^{\downarrow\blacklozenge})^{\uparrow\square}, \text{ by using (11)} \\
&\Longrightarrow P \subseteq (P^{\downarrow\blacklozenge})^{\uparrow\square}.
\end{aligned}$$

(P7):
i): Substituting P by $\widetilde{O}^{\uparrow\square}$ in (P6$'$) we get: $\widetilde{O}^{\uparrow\square} \subseteq ((\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square}$, and using (P2), we get : $(\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq (((\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge}$.
ii) Substituting $\widetilde{O}$ by $(\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge}$ in (P6), we get: $(((\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq (\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge}$.
From i) and ii), we conclude that: $(\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge} = (((\widetilde{O}^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge}$.
(P7$'$) : i) From the definitions (12) and (11), we have:

$$\begin{aligned}
&P^{\downarrow\blacklozenge} = \{o^{\alpha} \mid \alpha = \bigvee_{p\in P} \mathcal{I}(o,p)\} \\
&\Longrightarrow (P^{\downarrow\blacklozenge})^{\uparrow\square} = \{p \mid \forall o \in \mathcal{O}, \mathcal{I}(o,p) \leq P^{\downarrow\blacklozenge}(o)\} \\
&\Longrightarrow \begin{cases} \forall o \in \mathcal{O}, \forall p \in (P^{\downarrow\blacklozenge})^{\uparrow\square}, \mathcal{I}(o,p) \leq P^{\downarrow\blacklozenge}(o) \\ ((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge} = \{o^{\alpha} \mid \\ \qquad \alpha = \bigvee_{p\in((P^{\downarrow\blacklozenge})^{\uparrow\square})} \mathcal{I}(o,p)\} \end{cases} \\
&\Longrightarrow \forall o \in \mathcal{O}, ((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge}(o) = \\
&\qquad \bigvee_{p\in(P^{\downarrow\blacklozenge})^{\uparrow\square}} \mathcal{I}(o,p) \leq P^{\downarrow\blacklozenge}(o) \\
&\Longrightarrow ((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge} \subseteq P^{\downarrow\blacklozenge} \\
&\Longrightarrow (((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square} \subseteq (P^{\downarrow\blacklozenge})^{\uparrow\square}, \text{ by using (P1)}.
\end{aligned}$$

ii) Substituting P by $(P^{\downarrow\blacklozenge})^{\uparrow\square}$ in (P6$'$) (proved above) we get : $(P^{\downarrow\blacklozenge})^{\uparrow\square} \subseteq (((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square}$. From i) and ii) we conclude that : $(P^{\downarrow\blacklozenge})^{\uparrow\square} = (((P^{\downarrow\blacklozenge})^{\uparrow\square})^{\downarrow\blacklozenge})^{\uparrow\square}$ □

As a direct consequence of the Proposition 3, the Theorem 1 establishes that both $((.)^{\uparrow\square})^{\downarrow\blacklozenge}$ and $((.)^{\downarrow\blacklozenge})^{\uparrow\square}$ are order homomorphisms between the partially ordered sets $(L^{\mathcal{O}}, \subseteq)$ and $(2^{\mathcal{P}}, \subseteq)$. We recall hereafter the definition of an order homomorphism.

Definition 2. (Order homomorphism) *Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be partially ordered sets, and let $\varphi : A \to B$ be a mapping. φ is an order homomorphism if $\forall x, y \in A$, $x \preceq_A y \Rightarrow \varphi(x) \preceq_B \varphi(y)$.*

Theorem 1. *Let $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ be an L-context. The $\mathcal{FDM}$-compositions $((.)^{\uparrow\square})^{\downarrow\blacklozenge} : (L^{\mathcal{O}}, \subseteq) \to (L^{\mathcal{O}}, \subseteq)$ and $((.)^{\downarrow\blacklozenge})^{\uparrow\square} : (2^{\mathcal{P}}, \subseteq) \to (2^{\mathcal{P}}, \subseteq)$ are order homomorphisms.*

Proof.
The proof is easily established from (P5) and (P5′). □

Building on the above-established algebraic properties, namely properties P6, P6', P7 and P7', related to the fixed point theory, some particular pairs of $\langle objects, properties\rangle$ may be distinguished as relevant granules of knowledge on the meaning of (classical) formal concepts. These granules of knowledge are referred to as fuzzy dual-modal concepts ($\mathcal{FDM}$-concepts) and defined in the following:

Definition 3. ($\mathcal{FDM}$ − concept) *Let* $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ *be an L-context,* $\widetilde{O} \in L^{\mathcal{O}}$ *and* $P \in 2^{\mathcal{P}}$. *A pair* $\langle \widetilde{O}, P\rangle$ *s.t.* $\widetilde{O}^{\uparrow^{\Box}} = P$ *and* $P^{\downarrow^{\blacklozenge}} = \widetilde{O}$ *is a* $\mathcal{FDM}$*-concept.*

It may be remarked that, a $\mathcal{FDM}$-concept $\langle \widetilde{O}, P\rangle$ verifies $\widetilde{O} = (\widetilde{O}^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}$ and $P = (P^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}}$.

Example 3. *Consider the L-context given in Example 1. Then the pair* $\langle\{Paul^{0.9}, Sandy^{0.8}, Mike^{0.8}, Nelly^{0.4}\}, \{Python, C\#\}\rangle$ *is an* $\mathcal{FDM}$*-concept, since* $\{Paul^{0.9}, Sandy^{0.8}, Mike^{0.8}, Nelly^{0.4}\}^{\uparrow^{\Box}} = \{Python, C\#\}$ *and* $\{Python, C\#\}^{\downarrow^{\blacklozenge}} = \{Paul^{0.9}, Sandy^{0.8}, Mike^{0.8}, Nelly^{0.4}\}$.
By contrast, the pair $\langle\{Paul^{0.3}, Sandy^{0.8}, Mike^{0.8}, Nelly^{0.4}\}, \{C\#\}\rangle$ *is not an* $\mathcal{FDM}$*-concept, since* $\{Paul^{0.3}, Sandy^{0.8}, Mike^{0.8}, Nelly^{0.4}\}^{\uparrow^{\Box}} = \{C\#\}$ *and* $\{C\#\}^{\downarrow^{\blacklozenge}} = \{Paul^{0.2}, Sandy^{0.5}, Mike^{0.8}, Nelly^{0.1}\}$.

4 $\mathcal{FDM}$-concept lattice

This section establishes the lattice structure of $\mathcal{FDM}$-concepts via a fundamental theorem. It presents algorithms for computing these concepts and constructing the associated Hasse diagram, followed by an experimental evaluation.

4.1 Fundamental theorem

Let $\mathfrak{P}(\widetilde{\mathcal{K}})$ denote the set of all $\mathcal{FDM}$-concepts induced by the L-context $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$. The following lemma establishes two preliminary properties of $\mathfrak{P}(\widetilde{\mathcal{K}})$ that will be used in the proof of the fundamental theorem.

Lemma 1. *Let* $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ *be an L-context and* $\mathfrak{P}(\widetilde{\mathcal{K}})$ *be the set of all* $\mathcal{FDM}$*-concepts induced by* $\widetilde{\mathcal{K}}$. *Let* $E = \{\langle \widetilde{O}_i, P_i\rangle | i = 1, \dots, n\} \subseteq \mathfrak{P}(\widetilde{\mathcal{K}})$ *be an arbitrary family of* $\mathcal{FDM}$*-concepts. Then:*

$$(\mathrm{P8}): \left\langle \bigvee_{i=1..n} \widetilde{O}_i, \left(\left(\bigcup_{i=1..n} P_i\right)^{\downarrow^{\blacklozenge}}\right)^{\uparrow^{\Box}} \right\rangle \in \mathfrak{P}(\widetilde{\mathcal{K}})$$

$$(\mathrm{P9}): \left\langle \left(\left(\bigwedge_{i=1..n} \widetilde{O}_i\right)^{\uparrow^{\Box}}\right)^{\downarrow^{\blacklozenge}}, \bigcap_{i=1..n} P_i \right\rangle \in \mathfrak{P}(\widetilde{\mathcal{K}})$$

Proof. (P8): we have to prove that:

$$i)\ (\bigvee_{i=1..n} \widetilde{O}_i)^{\uparrow^{\Box}} = ((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}} \text{ and}$$

$$ii)\ \bigvee_{i=1..n} \widetilde{O}_i = (((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}.$$

$i)$ Since $\widetilde{O}_i = P_i^{\downarrow^{\blacklozenge}}$ (i.e. $\langle \widetilde{O}_i, P_i\rangle \in \mathfrak{P}(\widetilde{\mathcal{K}})$), and by using (P3'), we have:

$$\begin{aligned}(\bigvee_{i=1..n} \widetilde{O}_i)^{\uparrow^{\Box}} &= (\bigvee_{i=1..n} (P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}} \\ &= ((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}}.\end{aligned}$$

$ii)$ In one hand, we have:

$$\begin{aligned}&(((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}} \\ =\ &((\bigvee_{i=1..n} P_i^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}, \text{ by using (P3}')\\ =\ &((\bigvee_{i=1..n} \widetilde{O}_i)^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}, \text{ since } \widetilde{\mathrm{O}}_{\mathrm{i}} = \mathrm{P}_{\mathrm{i}}^{\downarrow^{\blacklozenge}}\\ \supseteq\ &(\bigcup_{i=1..n} \widetilde{O}_i^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}, \text{ by using (P3)}\\ =\ &(\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}}, \text{ since } \widetilde{\mathrm{O}}_{\mathrm{i}}^{\uparrow^{\Box}} = \mathrm{P}_{\mathrm{i}}\\ =\ &\bigvee_{i=1..n} P_i^{\downarrow^{\blacklozenge}}, \text{ by using (P3}')\\ =\ &\bigvee_{i=1..n} \widetilde{O}_i, \text{ since } \widetilde{\mathrm{O}}_{\mathrm{i}} = \mathrm{P}_{\mathrm{i}}^{\downarrow^{\blacklozenge}}.\end{aligned}$$

Then, we have:

$$\bigvee_{i=1..n} \widetilde{O}_i \subseteq (((\bigcup_{=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}.$$

In other hand, From the property (P6) (cf. Proposition 3), we get:

$$\begin{aligned}&((\bigvee_{i=1..n} \widetilde{O}_i)^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}} \subseteq \bigvee_{i=1..n} \widetilde{O}_i\\ \Longrightarrow\ &((\bigvee_{i=1..n} P_i^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}} \subseteq \bigvee_{i=1..n} \widetilde{O}_i \text{ , since } \widetilde{\mathrm{O}}_{\mathrm{i}} = \mathrm{P}_{\mathrm{i}}^{\downarrow^{\blacklozenge}}\\ \Longrightarrow\ &(((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}} \subseteq \bigvee_{i=1..n} \widetilde{O}_i \text{ , using (P3}').\end{aligned}$$

It follows that:

$$\bigvee_{i=1..n} \widetilde{O}_i = (((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}})^{\downarrow^{\blacklozenge}}$$

From $i)$ and $ii)$ we deduce that:

$$\langle \bigvee_{i=1..n} \widetilde{O}_i, ((\bigcup_{i=1..n} P_i)^{\downarrow^{\blacklozenge}})^{\uparrow^{\Box}}\rangle \in \mathfrak{P}(\widetilde{\mathcal{K}})$$

The proof of Property (P9) relies on reasoning analogous to that of Property (P8) and is therefore omitted here. □

We consider now a partial order (denoted $\ll$) on the set $\mathfrak{P}(\widetilde{\mathcal{K}})$, defined as $\langle \widetilde{O}_1, P_1\rangle \ll \langle \widetilde{O}_2, P_2\rangle$ iff $\widetilde{O}_1 \subseteq \widetilde{O}_2$, or equivalently, $P_1 \subseteq P_2$ (it is easy to prove that $\ll$ is a partial order). The following fundamental theorem establishes an important result by giving the algebraic structure of the set $\mathfrak{P}(\widetilde{\mathcal{K}})$ equipped with the partial order $\ll$.

Theorem 2. *Let $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ be an L-context. The set $\mathfrak{P}(\widetilde{\mathcal{K}})$ ordered with $\ll$ is a complete lattice in which supremum and infimum are given by:*

$$\bigvee_{i\in I} \left\langle \widetilde{O}_i, P_i \right\rangle = \left\langle \bigvee_{i\in I} \widetilde{O}_i, \left(\left(\bigcup_{i\in I} P_i\right)^{\downarrow^\blacklozenge}\right)^{\uparrow^\square} \right\rangle \quad (13)$$

$$\bigwedge_{i\in I} \left\langle \widetilde{O}_i, P_i \right\rangle = \left\langle \left(\left(\bigwedge_{i\in I} \widetilde{O}_i\right)^{\uparrow^\square}\right)^{\downarrow^\blacklozenge}, \bigcap_{i\in I} P_i \right\rangle \quad (14)$$

Proof.
Supremum (13):
From the Lemma 1, we deduce that:

$$\langle \bigvee_{i\in I} \widetilde{O}_i, ((\bigcup_{i=1\in I} P_i)^{\downarrow^\blacklozenge})^{\uparrow^\square} \rangle \in \mathfrak{P}(\widetilde{\mathcal{K}})$$

Let us denote it by $\langle \widetilde{O}_s, P_s\rangle$). Thus, we have just to prove that $\langle \widetilde{O}_s, P_s\rangle$ is the least upper bound of the set $\{\langle \widetilde{O}_i, P_i\rangle, \, i \in I\}$.
i) We have: $\forall i \in I, \widetilde{O}_i \leq \bigvee_{i\in I} \widetilde{O}_i$ and $P_i \subseteq \bigcup_{i\in I} P_i \subseteq \left(\left(\bigcup_{i\in I} P_i\right)^{\downarrow^\blacklozenge}\right)^{\uparrow^\square}$ (by using (P3')). Thus, $\forall i \in I$ $\langle \widetilde{O}_i, P_i\rangle \ll \langle \widetilde{O}_s, P_s\rangle$ and consequently $\langle \widetilde{O}_s, P_s\rangle$ is an upper bound of $\langle \widetilde{O}_i, P_i\rangle$.
ii)Let us suppose there exists a $\mathcal{FDM}$-concept $\langle \widetilde{O}_l, P_l\rangle$ ($l \neq s$) which is the least upper bound of the set $\{\langle \widetilde{O}_i, P_i\rangle, \, i \in I\}$, then we have:

$$\begin{aligned}
\forall i \in I, \quad & \langle \widetilde{O}_i, P_i\rangle \ll \langle \widetilde{O}_l, P_l\rangle \\
& \Longrightarrow \begin{cases} \widetilde{O}_i \subseteq \widetilde{O}_l, \forall i \in I \\ P_i \subseteq P_l, \forall i \in I \end{cases} \\
& \Longrightarrow \begin{cases} \bigvee_{i\in I} \widetilde{O}_i \subseteq \widetilde{O}_l \\ \bigcup_{i\in I} P_i \subseteq P_l \end{cases} \\
& \Longrightarrow \begin{cases} \bigvee_{i\in I} \widetilde{O}_i \subseteq \widetilde{O}_l \\ \left(\bigcup_{i\in I} P_i\right)^{\downarrow^\blacklozenge} \subseteq P_l^{\downarrow^\blacklozenge}, \text{ by using (P2)} \end{cases} \\
& \Longrightarrow \begin{cases} \bigvee_{i\in I} \widetilde{O}_i \subseteq \widetilde{O}_l \\ \left(\left(\bigcup_{i\in I} P_i\right)^{\downarrow^\blacklozenge}\right)^{\uparrow^\square} \subseteq \left(P_l^{\downarrow^\blacklozenge}\right)^{\uparrow^\square}, \\ \qquad \text{by using (P1)} \end{cases} \\
& \Longrightarrow \begin{cases} \widetilde{O}_s \subseteq \widetilde{O}_l \,, \text{ since } \bigvee_{\mathrm{i}\in \mathrm{I}} \widetilde{\mathrm{O}}_\mathrm{i} = \widetilde{\mathrm{O}}_\mathrm{s} \\ P_s \subseteq P_l, \text{since } ((\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{P_i})^{\downarrow^\blacklozenge})^{\uparrow^\square} = \mathrm{P_s} \end{cases} \\
& \Longrightarrow \langle \widetilde{O}_s, P_s\rangle \ll \langle \widetilde{O}_l, P_l\rangle.
\end{aligned}$$

From i) and ii), it follows that $\langle \widetilde{O}_s, P_s\rangle$ is the least upper bound of the set $\{\langle \widetilde{O}_i, P_i\rangle, \, i \in I\}$

The expression of the infimum (14) is proved using reasoning analogous to that of the expression of the supremum (13) and is omitted here for conciseness. □

4.2 Lattice construction

Given an L-context $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$, let $\mathfrak{P}(\widetilde{\mathcal{K}})$ denote the set of all $\mathcal{FDM}$-concepts induced by $\widetilde{\mathcal{K}}$.

Intuitively, as described in Algorithm 1, the set $\mathfrak{P}(\widetilde{\mathcal{K}})$ is constructed by enumerating all subsets of $\mathcal{P}$. For each subset $P \subseteq \mathcal{P}$, the fuzzy set of objects $\widetilde{O} = P^{\downarrow^\blacklozenge}$ is first computed, followed by the property set $P' = \widetilde{O}^{\uparrow^\square}$. If $P' = P$, then the pair $\langle \widetilde{O}, P\rangle$ forms a new $\mathcal{FDM}$-concept and is added to $\mathfrak{P}(\widetilde{\mathcal{K}})$.

Algorithm 1 Compute-$\mathfrak{P}(\widetilde{\mathcal{K}})$: Powerset-Based Algorithm

Input: $\widetilde{\mathcal{K}}$ ▷ Fuzzy context $(L, \mathcal{O}, \mathcal{P}, \mathcal{I})$
Output: $\mathfrak{P}(\widetilde{\mathcal{K}})$ ▷ The set of all $\mathcal{FDM}$-concepts

```
1: for each P ∈ 2^𝒫 do
2:     Õ ← P^{↓♦}
3:     P′ ← Õ^{↑□}
4:     if P′ == P then
5:         𝔓(K̃) ← 𝔓(K̃) ⋃ {⟨Õ, P⟩}
6:     end if
7: end for
```

Let $n = |\mathcal{O}|$ and $m = |\mathcal{P}|$ denote the numbers of objects and properties, respectively. The time complexity of the proposed algorithm is $\mathcal{O}(2^m \cdot n \cdot m)$. The main loop (Line 1, Algorithm 1) enumerates all subsets of properties and thus runs in $\mathcal{O}(2^m)$. For each subset, computing the fuzzy set of objects (Line 2) and the associated property set (Line 3) each requires $\mathcal{O}(n \cdot m)$.

Several attribute subsets may generate the same $\mathcal{FDM}$-concept, leading to redundant computations since only one subset uniquely defines it. For example, in the fuzzy formal context of Table 1, both $\{Java, C\#\}$ and $\{Java\}$ produce the $\mathcal{FDM}$-concept $\langle\{Paul^{0.7}, Sandy^{0.7}, Mike^{1.0}, Nelly^{0.6}\}, \{Java, C\#\}\rangle$. To improve time complexity, $\mathcal{FDM}$-concepts can be generated incrementally by exploring property combinations in lexicographical order, thereby reducing redundant computations. Let $<$ be a lexicographical order on $\mathcal{P}$ such that $p_1 < p_2 < \cdots < p_m$. The recursive procedure in Algorithm 2 generates new $\mathcal{FDM}$-concepts from previously computed one by adding appropriate properties. At the initial stage, the set $\mathfrak{P}(\widetilde{\mathcal{K}})$ is initialized with the bottom element $\langle \emptyset^{\downarrow^\blacklozenge}, \emptyset\rangle$, where $\emptyset^{\downarrow^\blacklozenge}$ denotes the fuzzy set of objects with null membership degrees. At each subsequent stage, the recursive algorithm selects properties according to the lexicographical order and calls the procedure $Generate()$ (described in Algorithm 3). The procedure $Generate()$ incrementally extends the current $\mathcal{FDM}$-concept by adding lexicographically greater properties to compute new potential concepts.

Algorithm 2 Compute-$\mathfrak{P}(\widetilde{\mathcal{K}})$: Recursive Algorithm

Input: $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$
Output: $\mathfrak{P}(\widetilde{\mathcal{K}})$ ▷ The set of all $\mathcal{FDM}$-concepts generated from $\widetilde{\mathcal{K}}$

1: $\mathfrak{P}(\widetilde{\mathcal{K}}) = \{ \langle \emptyset^{\downarrow^{\blacklozenge}}, \emptyset \rangle \}$
2: **for each** $p_i \in \mathcal{P}$ **do**
3:     $A \leftarrow \{p_i\}$
4:     Generate(A , p_i , $\langle \{p_i\}^{\downarrow^{\blacklozenge}}, (\{p_i\}^{\downarrow^{\blacklozenge}})^{\uparrow^{\square}} \rangle$)
5: **end for**

The pseudocode of the procedure $Generate()$ described in Algorithm 3 is explained hereafter. For more readability, we consider subsets $A, B \in 2^{\mathcal{P}}$ and $\widetilde{X}, \widetilde{Y} \in L^{\mathcal{O}}$:

Line 1- Verify that the $\mathcal{FDM}$-concept $\langle \widetilde{X}, B \rangle$ has not been generated previously. A previously generated concept is detected by the presence, in the set B, of properties that are lexicographically smaller than the current property p_i and that do not belong to the set A. In such cases, no further recursion is performed, and the recursive call is skipped. The set A is used to keep track of candidate properties that have already been visited and that are lexicographically greater than the initial property considered in Algorithm 2-line 3.

Algorithm 3 Generate(A , p_i , $\langle \widetilde{X}, B \rangle$)

1: **if** $\{p_j | p_j \in B \ and \ p_j \notin A \ and \ p_j < p_i\} == \emptyset$ **then**
2:     $\mathfrak{P} \leftarrow \mathfrak{P} \bigcup \{ \langle \widetilde{X}, B \rangle \}$
3:     **for each** $p \in \{p_k | p_k \in \mathcal{P} \ and \ p_k > p_i\}$ **do**
4:         $A \leftarrow A \bigcup \{p\}$
5:         $\widetilde{Y} \leftarrow (B \bigcup \{p\})^{\downarrow^{\blacklozenge}}$
6:         $B \leftarrow \widetilde{Y}^{\uparrow^{\square}}$
7:         Generate(A , p , $\langle \widetilde{Y}, B \rangle$)
8:     **end for**
9: **end if**

Line 2- adds $\langle \widetilde{X}, B \rangle$ to $\mathfrak{P}(\widetilde{\mathcal{K}})$ since $\langle \widetilde{X}, B \rangle \notin \mathfrak{P}(\widetilde{\mathcal{K}})$;
Line 3- Iterate over properties that are lexicographically greater than p_i.
Line 4- adds the candidate property to the subset A;
Line 5- computes the crisp set of properties of a new candidate $\mathcal{FDM}$-concept;
Line 6- computes the fuzzy set of objects of a new candidate $\mathcal{FDM}$-concept;
Line 7- calls the recursive procedure $Generate()$ with the new parameters.

The incremental algorithm runs in $\mathcal{O}(c \cdot n \cdot m)$ time, where $c = |\mathfrak{P}(\widetilde{\mathcal{K}})|$ denotes the number of induced $\mathcal{FDM}$-concepts. Hence, the algorithm is output-polynomial in the number of generated concepts. In the worst case, since c may grow exponentially with respect to the number of attributes m, the overall complexity becomes $\mathcal{O}(2^m \cdot n \cdot m)$.

Despite the exponential worst-case bound, the algorithm is efficient in practice as it is output-polynomial. This efficiency results from pruning recursive calls in $Generate()$ that do not satisfy the condition in Line 1 of Algorithm 3, thereby restricting the search space and making the cost proportional to the number of generated concepts (see Subsection 4.3). The recursive generation of $\mathcal{FDM}$-concepts ensures that each concept is produced exactly once, avoiding redundant exploration of the property powerset. Further improvements may be obtained by memoizing the intermediate results of Lines 5–6 in Algorithm 3 and exploiting Properties (P3′) and (P4) (Proposition 2), which reduce repeated computations in large contexts. Additional efficiency may also be achieved using compact bitset representations for property sets, enabling fast set operations. While the recursion depth is bounded by m, the overall scalability mainly depends on the number of generated concepts c, which remains relatively low within the proposed $\mathcal{FDM}$ framework.

To construct the Hasse diagram of the $\mathcal{FDM}$-concept lattice, each concept in $\mathfrak{P}(\widetilde{\mathcal{K}})$ is represented as a node, and edges encode the partial order relation $\ll$. The Hasse diagram encodes only the covering relation of this order. Given two $\mathcal{FDM}$-concepts f_1 and f_2, f_2 covers f_1 if $f_1 \ll f_2$ and there exists no $\mathcal{FDM}$-concept f such that $f_1 \ll f \ll f_2$. This yields a concise visualization of the lattice while preserving the structure of the partial order. Algorithm 4 constructs the Hasse diagram by examining all pairs $(f_1, f_2) \in \mathfrak{P}(\widetilde{\mathcal{K}}) \times \mathfrak{P}(\widetilde{\mathcal{K}})$ and identifying those such that f_2 covers f_1.

Algorithm 4 Construction of the Hasse Diagram from $\mathfrak{P}(\widetilde{\mathcal{K}})$

Input: $\mathfrak{P}(\widetilde{\mathcal{K}})$
Output: $H = (N, E)$ ▷ The Hasse diagram, ▷ N the set of nodes, E the set of edges

$N \leftarrow \mathfrak{P}(\widetilde{\mathcal{K}})$ ▷ Each concept becomes a node
$E \leftarrow \emptyset$ ▷ Initialize empty edge set
for all $f_1 \in \mathfrak{P}(\widetilde{\mathcal{K}})$ **do**
    for all $f_2 \in \mathfrak{P}(\widetilde{\mathcal{K}})$ **do**
        if $f_1 \ll f_2$ **then**
            isCovered $\leftarrow$ **true**
            for all $f \in \mathfrak{P}(\widetilde{\mathcal{K}})$ **do**
                if $f_1 \ll f \ \wedge \ f \ll f_2$ **then**
                    isCovered $\leftarrow$ **false**
                    break
                end if
            end for
            if isCovered **then**
                $E \leftarrow E \cup \{(f_1, f_2)\}$
            end if
        end if
    end for
end for

Let $c = |\mathfrak{P}(\widetilde{\mathcal{K}})|$ denote the number of $\mathcal{FDM}$-concepts. To construct the Hasse diagram, the algorithm compares all pairs of concepts to detect cover relations, requiring $\mathcal{O}(c^2)$ operations. For each pair, it may further check whether an intermediate concept exists, leading to a worst-case time complexity of $\mathcal{O}(c^3)$. Although the worst-case complexity is cubic, the practical cost can be reduced by restricting comparisons to structurally comparable concept pairs and applying early stopping in transitive reduction once a covering relation is found. Moreover, the number of concepts generated by the proposed $\mathcal{FDM}$ framework is typically lower than in classical fuzzy FCA, making Hasse diagram construction feasible for moderately sized datasets, while larger instances may require additional optimizations or specialized data structures.

4.3 Experimental comparison

In this subsection, we present an experimental evaluation of the proposed $\mathcal{FDM}$ framework. The objective is to analyze its computational performance and to examine the size and structural properties of the induced concept lattices, in comparison with fuzzy FCA and (crisp) modal FCA approaches.

We evaluated the two proposed $\mathcal{FDM}$ algorithms, namely the powerset-based and recursive variants, together with two fuzzy FCA baselines, Next-Closure and In-Close, implemented using the Łukasiewicz implication and the finite scale $L = \{0, 0.1, \dots, 0.9, 1\}$. In addition, the recursive $\mathcal{FDM}$ algorithm was adapted to compute (crisp) modal concepts. All algorithms were implemented in `C++` to ensure consistent and reproducible experimental comparisons.[1]

The recursive $\mathcal{FDM}$ algorithm was implemented in two variants. The first corresponds to a direct and faithful implementation of Algorithm 2, in which attribute sets are represented using ordered set structures. The second variant preserves the underlying computational logic while incorporating several optimizations, including a compact representation of attribute sets and memoization of intermediate computations.

For comparison with (crisp) modal FCA, the fuzzy context $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ is converted into a crisp context $\mathcal{K}' = (\mathcal{O}, \mathcal{P}', \mathcal{R})$ via scaling techniques, namely a threshold-based strategy and a discretization-based strategy. In the threshold-based strategy, a threshold $\theta \in [0, 1]$ is introduced, and the fuzzy relation $\mathcal{I}$ is transformed into a binary relation $\mathcal{R}$ defined by $\mathcal{R}(o, p) = 1$ if $\mathcal{I}(o, p) \geq \theta$, and 0 otherwise. In the discretization-based strategy, for each property $p \in \mathcal{P}$, let $\{v_1, \dots, v_j\} = \{\mathcal{I}(o, p) \mid o \in \mathcal{O}\}$ denote the distinct satisfaction degrees of p in $\widetilde{\mathcal{K}}$. The property p is replaced in $\mathcal{K}'$ by j mutually exclusive properties $\{p'_1, \dots, p'_j\}$, where an object o satisfies p'_i (i.e., $\mathcal{R}(o, p'_i) = 1$) iff $\mathcal{I}(o, p) = v_i$.

The experimental evaluation considers two criteria: (i) the number of generated concepts and (ii) the runtime. Performance was assessed by increasing the number of attributes in randomly generated fuzzy formal contexts while keeping the number of objects fixed; Table 5 reports the corresponding execution times and concept counts for all considered algorithms.

Figure 1 presents runtime and lattice size for randomly generated fuzzy contexts on a logarithmic scale. Across all tested configurations, the $\mathcal{FDM}$ approach generates a significantly reduced number of concepts compared with fuzzy FCA and (crisp) modal FCA using the discretization-based strategy, while achieving lower execution times. Although threshold-based modal FCA may yield smaller lattices by discarding the graded information of the fuzzy context, the proposed $\mathcal{FDM}$ framework yields compact lattices while preserving the graded semantics of the original context, thereby providing an efficient means to control concept explosion.

To further assess the structural properties of concept lattices on real-world data, the proposed $\mathcal{FDM}$ framework is evaluated on three benchmark datasets: *Mushroom* [33], *Parkinsons* [34], and *Diabetes* [35]. The corresponding fuzzy formal contexts have dimensions 8124×22, 195×22, and 69×20, respectively. However, due to the high computational cost of fuzzy formal concept extraction on large datasets, the fuzzy FCA experiments on the *Mushroom* dataset are conducted using only 50% of the objects. The corresponding results are reported in Table 6.

A direct comparison with crisp-modal FCA is not considered. Although the threshold-based strategy reduces the number of concepts, it does so through discretization, which discards the gradual information inherent in fuzzy contexts. Conversely, the discretization-based approach enlarges the set of properties, resulting in a combinatorial growth in the number of concepts and a higher computational cost. Consequently, this study focuses on lattices generated directly in the fuzzy setting.

To characterize lattice size, connectivity, and structural complexity, several structural metrics are reported for each dataset. Specifically, we consider the number of nodes $|V|$, the number of arcs $|E|$, the average degree $\overline{d} = |E|/|V|$, as well as the height (length of the longest chain) and the width (size of a maximum antichain). For fuzzy FCA, these metrics are computed on the Hasse diagram of a partial concept lattice containing about 0.5%–10% of the concepts, as constructing the complete fuzzy concept lattice for large real-world datasets is computationally prohibitive.

The results in Table 6 show that fuzzy FCA generates lattices that are several orders of magnitude larger than those produced by $\mathcal{FDM}$, even though the reported values correspond to partial lattices containing only about 0.5%–10% of the concepts. While the average degree $\overline{d}$ remains comparable across both approaches, fuzzy FCA lattices exhibit substantially larger heights and widths, indicating a higher structural complexity.

5 $\mathcal{FDM}$-concepts applications

This section presents the application of the proposed approach to information retrieval (*IR* for short), disjunctive

[1] The source code is available at: `https://github.com/zerarga-loutfi/FDM/`.

Table 5: Comparison of concept counts and execution times (in seconds) for (crisp) modal FCA (threshold and discretization-based strategies), Fuzzy FCA (Next-Closure and In-Close), and $\mathcal{FDM}$ algorithms on randomly generated formal contexts with fixed $n = 25$ and varying m.

FCA Approach	Metric	25×10	25×15	25×20	25×25
Modal (threshold-based) $\theta = 0.7$	Concept counts	290	1558	2988	8206
	Execution time (s)	0.001	0.009	0.037	0.165
Modal (discretization-based)	New properties counts	86	120	171	201
	Concept counts	18016837	20709459	30454917	32385048
	Execution time (s)	7169.393	19588.790	61765.122	122238.523
Fuzzy	Concept counts	1626423	45597938	710778511	2674022918
	Next-Closure (s)	21.861	1625.267	27461.713	237030.663
	In-Close (s)	14.317	1014.671	17026.612	152923.009
$\mathcal{FDM}$	Concept counts	874	11229	148064	1262065
	Powerset (s)	0.003	0.130	5.415	187.155
	Recursive (s)	0.003	0.080	1.808	25.910
	Optimized Recursive (s)	0.001	0.047	0.794	7.601

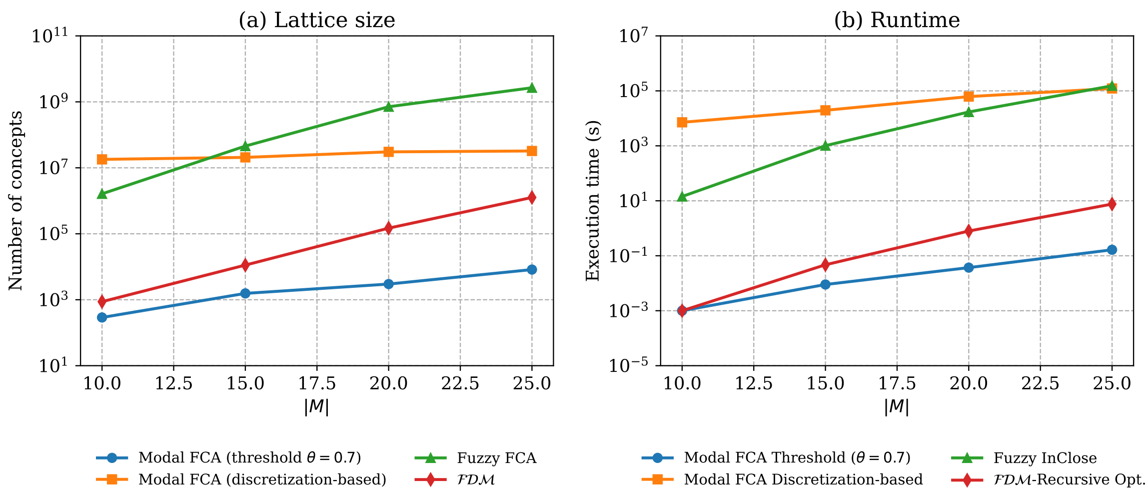

Figure 1: Evolution of (a) lattice size and (b) runtime with respect to the size of $\mathcal{P}$ for $\mathcal{FDM}$, (crisp) modal, and fuzzy FCA approaches.

attribute implications in fuzzy context, and adaptive and nonlinear control systems, assessing its generality across distinct application domains.

5.1 Disjunctive queries in IR

Let $\mathcal{D} = \{d_i \mid i = 1, \ldots, n\}$ be a set of documents and $\mathcal{T} = \{t_j \mid j = 1, \ldots, m\}$ a set of index terms. The application of the FCA framework to IR is motivated by the similarity between the incidence matrix $\mathcal{D} \times \mathcal{T}$ in IR and the $Objects \times Properties$ relation in FCA. In this context, Godin et al. [36] highlighted the relevance of the classical sufficiency operator, according to which the extent D of a formal concept $\langle D, T \rangle$ can be interpreted as the set of documents satisfying a conjunctive query composed of the terms in T.

By exploiting the semantics of both $\mathcal{FM}$ operators $(.)^{\downarrow^\blacklozenge}$ and $(.)^{\uparrow^\square}$, the $\mathcal{FDM}$ approach enables the expression of disjunctive queries rather than conjunctive ones. In this setting, an $\mathcal{FDM}$-concept $\langle \widetilde{D}, T \rangle$ can be interpreted as a pair $\langle \mathit{Answer}, \mathit{Query} \rangle$, where the fuzzy set of documents $\widetilde{D}$ represents the answer to the disjunction of the terms in T. Moreover, the neighbourhood structure of $\mathcal{FDM}$-concepts in the lattice naturally supports query refinement and enlargement.

As an illustrative example, consider the L-context (incidence matrix) shown in Table 7, which associates with each pair $(d_i, t_j) \in \mathcal{D} \times \mathcal{T}$ the weight of term t_j in document d_i, taking values in $[0, 1]$.

The retrieval process consists of generating all $\mathcal{FDM}$-concepts of the form $\langle \widetilde{D}, T \rangle$, where the fuzzy set $\widetilde{D} \in [0, 1]^{\mathcal{D}}$ corresponds to the answer set of the disjunctive query defined by T $(t_1 \vee t_2 \vee \cdots \vee t_k)$. The collection of these $\mathcal{FDM}$-concepts is organized as the lattice shown

Table 6: Structural comparison of concept lattices induced by $\mathcal{FDM}$ and fuzzy FCA approaches on real-world datasets.

Dataset	Metric	Approach	
		$\mathcal{FDM}$	Fuzzy FCA
Mushroom	Concept counts	81534	13791006
	$\lvert V\rvert$	81534	1379100
	$\lvert E\rvert$	533411	9086749
	d	6.54	6.58
	Height	23	82
	Width	11945	94046
Parkinsons	Concept counts	379534	1153862353
	$\lvert V\rvert$	379534	5769312
	$\lvert E\rvert$	3188332	39802170
	d	8.40	6,89
	Height	21	72
	Width	62143	208857
Diabetes	Concept counts	500096	44313594
	$\lvert V\rvert$	500096	4431360
	$\lvert E\rvert$	4518784	29450623
	d	9.03	6.64
	Height	21	68
	Width	82640	206300

in Figure 2.

Table 7: Example of Incidence Matrix $\mathcal{D} \times \mathcal{T}$ Related to IR.

$\mathcal{I}$	t_1	t_2	t_3	t_4	t_5
d_1	0.4	0.1	0.2	0.4	0.3
d_2	0.7	0.1	0.5	0.1	0.3
d_3	0.8	0.2	0.2	0.2	0.5
d_4	0.2	0.1	0.5	0.1	0.2
d_5	0.4	0.2	0.4	0.5	0.3
d_6	0.6	0.2	0.2	0.4	0.1

The proposed $\mathcal{FDM}$-based query model is evaluated on three IR benchmarks (DBpedia-Entity [37], NFCorpus [38], and WikIR [39]) and compared against classical bag-of-words term-matching retrieval models. The incidence matrix $\mathcal{D}\times\mathcal{T}$ is computed using BM25-style weighting [40], combining term-frequency (TF) saturation and inverse document frequency (IDF) weighting[2]. Comparisons are conducted with BM25 [40], the TF-weighted vector space model with cosine similarity, Term-Frequency sum (TF-Sum), and Boolean OR/AND matching [41]. Evaluation relies on macro-averaged Precision@k, Recall@k, and F_1@k for $k \in 10, 100$. As reported in Table 8, the proposed $\mathcal{FDM}$-based query model performs comparably to the considered lexical baselines, achieving competitive F_1 scores and consistently high recall, particularly at $k = 100$.

[2]BM25 parameters are set to k_1=1.2 and b=0.75.

5.2 Disjunctive attribute implications

Disjunctive attribute implications express dependencies between subsets of attributes. They mainly rely on works pertaining to functional dependencies in database theory [8]. Let $\mathcal{K} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ be a formal context, let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be two subsets of $\mathcal{P}$. A disjunctive attribute implication $\bigvee A \mapsto \bigvee B$ (or $a_1 \vee a_2 \vee \cdots \vee a_n \mapsto b_1 \vee b_2 \vee \cdots \vee b_m$) is valid in $\mathcal{K}$ (i.e. $\mathcal{K} \models \bigvee A \mapsto \bigvee B$), when for each object $o \in \mathcal{O}$, if there exists a_i s.t. $(x, a_i) \in \mathcal{R}$, then exist b_i s. t. $(x, b_i) \in \mathcal{R}$ [8].

It is worth noting that, in order to extract such dependencies, the authors in [8] proposed a dual logic based on the dual formal context (i.e., $\overline{\mathcal{K}} = (\mathcal{O}, \mathcal{P}, \overline{\mathcal{R}})$, where $\overline{\mathcal{R}}$ denotes the complement of $\mathcal{R}$). However, this approach is limited to disjunctive attribute implications extracted from non-fuzzy formal contexts. In this work, we consider disjunctive attribute implications directly over fuzzy formal contexts, as defined below.

Definition 4. ($Disjunctive\ attribute\ implication$) *Given an L-context $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$, and $A, B \in 2^{\mathcal{P}}$, a disjunctive attribute implication $A \rightarrowtail B$ is valid in $\widetilde{\mathcal{K}}$ (i.e. $\widetilde{\mathcal{K}} \models A \rightarrowtail B$), if $\forall o \in \mathcal{O}$, if there exists $a_i \in A$ such that $\mathcal{I}(o, a_i) \leq \alpha$, then there exists $b_j \in B$ such that $\mathcal{I}(o, b_j) \leq \beta$ and $0 \leq \alpha \leq \beta \leq 1$.*

Proposition 4. *Let $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$ be an L-context, and let $A, B \in 2^{\mathcal{P}}$ then:*

$$\widetilde{\mathcal{K}} \models A \rightarrowtail B \Longleftrightarrow A^{\downarrow\blacklozenge} \subseteq B^{\downarrow\blacklozenge}$$

Proof. We have to prove that:

$$i) \qquad \widetilde{\mathcal{K}} \models A \rightarrowtail B \Longrightarrow A^{\downarrow\blacklozenge} \subseteq B^{\downarrow\blacklozenge} \text{ and}$$

$$ii) \qquad A^{\downarrow\blacklozenge} \subseteq B^{\downarrow\blacklozenge} \Longrightarrow \widetilde{\mathcal{K}} \models A \rightarrowtail B$$

$i)$ from the Definition 4, we have:

$$\begin{aligned}
&\widetilde{\mathcal{K}} \models A \rightarrowtail B \\
&\Longrightarrow \forall o \in \mathcal{O}, \forall a \in A, \forall b \in B,\ \mathcal{I}(o, a) \leq \mathcal{I}(o, b) \\
&\Longrightarrow \forall o \in \mathcal{O}, \bigvee_{a \in A} \mathcal{I}(o, a) \leq \bigvee_{b \in B} \mathcal{I}(o, b) \\
&\Longrightarrow A^{\downarrow\blacklozenge} \subseteq B^{\downarrow\blacklozenge},\ \text{ by using (12).}
\end{aligned}$$

$$\begin{aligned}
ii) \qquad & A^{\downarrow\blacklozenge} \subseteq B^{\downarrow\blacklozenge} \\
&\Longrightarrow \forall o \in \mathcal{O}, \bigvee_{a_i \in A} \mathcal{I}(o, a_i) \leq \bigvee_{b_j \in B} \mathcal{I}(o, b_j) \\
&\Longrightarrow \forall o \in \mathcal{O}, \forall a \in A, \forall b \in B, \\
&\qquad \begin{cases} \mathcal{I}(o, a) \leq \bigvee_{a_i \in A} \mathcal{I}(o, a_i) \\ \mathcal{I}(o, b) \leq \bigvee_{b_j \in B} \mathcal{I}(o, b_j) \\ \bigvee_{a_i \in A} \mathcal{I}(o, a_i) \leq \bigvee_{b_j \in B} \mathcal{I}(o, b_j) \end{cases} \\
&\Longrightarrow \widetilde{\mathcal{K}} \models A \rightarrowtail B,\ \text{by taking}
\end{aligned}$$

$\alpha = \bigvee_{a_i \in A} \mathcal{I}(o, a_i)$ and $\beta = \bigvee_{\mathrm{b_j} \in \mathrm{B}} \mathcal{I}(\mathrm{o}, \mathrm{b_j})$. ☐

ID	$\langle D, T\rangle$
C_0	$\langle\{d_1^{0.0}, d_2^{0.0}, d_3^{0.0}, d_4^{0.0}, d_5^{0.0}, d_6^{0.0}\}, \emptyset\rangle$
C_1	$\langle\{d_1^{0.1}, d_2^{0.1}, d_3^{0.2}, d_4^{0.1}, d_5^{0.2}, d_6^{0.2}\}, \{t_2\}\rangle$
C_2	$\langle\{d_1^{0.3}, d_2^{0.3}, d_3^{0.5}, d_4^{0.2}, d_5^{0.3}, d_6^{0.1}\}, \{t_5\}\rangle$
C_3	$\langle\{d_1^{0.2}, d_2^{0.5}, d_3^{0.2}, d_4^{0.5}, d_5^{0.4}, d_6^{0.2}\}, \{t_2, t_3\}\rangle$
C_4	$\langle\{d_1^{0.4}, d_2^{0.1}, d_3^{0.2}, d_4^{0.1}, d_5^{0.5}, d_6^{0.4}\}, \{t_2, t_4\}\rangle$
C_5	$\langle\{d_1^{0.3}, d_2^{0.3}, d_3^{0.5}, d_4^{0.2}, d_5^{0.3}, d_6^{0.2}\}, \{t_2, t_5\}\rangle$
C_6	$\langle\{d_1^{0.4}, d_2^{0.5}, d_3^{0.2}, d_4^{0.5}, d_5^{0.5}, d_6^{0.4}\}, \{t_2, t_3, t_4\}\rangle$
C_7	$\langle\{d_1^{0.3}, d_2^{0.5}, d_3^{0.5}, d_4^{0.5}, d_5^{0.4}, d_6^{0.2}\}, \{t_2, t_3, t_5\}\rangle$
C_8	$\langle\{d_1^{0.4}, d_2^{0.3}, d_3^{0.5}, d_4^{0.2}, d_5^{0.5}, d_6^{0.4}\}, \{t_2, t_4, t_5\}\rangle$
C_9	$\langle\{d_1^{0.4}, d_2^{0.7}, d_3^{0.8}, d_4^{0.2}, d_5^{0.4}, d_6^{0.6}\}, \{t_1, t_2, t_5\}\rangle$
C_{10}	$\langle\{d_1^{0.4}, d_2^{0.5}, d_3^{0.5}, d_4^{0.5}, d_5^{0.5}, d_6^{0.4}\}, \{t_2, t_3, t_4, t_5\}\rangle$
C_{11}	$\langle\{d_1^{0.4}, d_2^{0.7}, d_3^{0.8}, d_4^{0.5}, d_5^{0.4}, d_6^{0.6}\}, \{t_1, t_2, t_3, t_5\}\rangle$
C_{12}	$\langle\{d_1^{0.4}, d_2^{0.7}, d_3^{0.8}, d_4^{0.2}, d_5^{0.5}, d_6^{0.6}\}, \{t_1, t_2, t_4, t_5\}\rangle$
C_{13}	$\langle\{d_1^{0.4}, d_2^{0.7}, d_3^{0.8}, d_4^{0.5}, d_5^{0.5}, d_6^{0.6}\}, \{t_1, t_2, t_3, t_4, t_5\}\rangle$

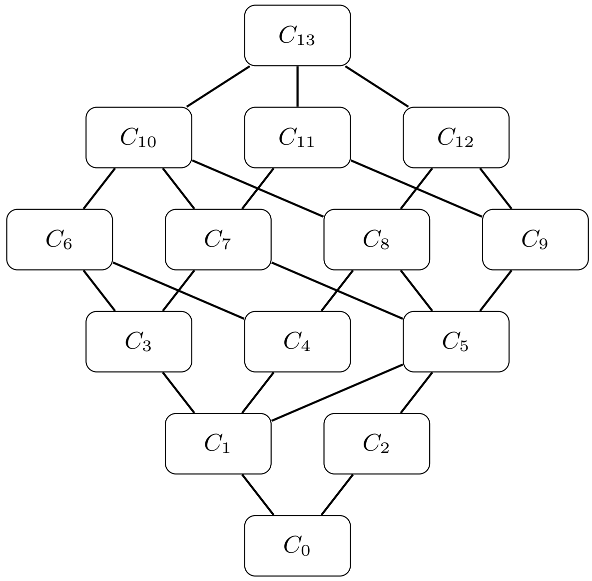

Figure 2: The $\mathcal{FDM}$-concepts induced by the L-context of Table 7 (left) and the corresponding Hasse diagram (right). Each node C_i represents an $\mathcal{FDM}$-concept $\langle\widetilde{D}, T\rangle$, where the fuzzy set $\widetilde{D}$ denotes the answer to the disjunctive query defined by the set of terms T. Edges encode the partial order $\ll$. Moving upward in the lattice corresponds to query expansion, whereas moving downward corresponds to query restriction.

Table 8: Macro-averaged Precision (P), Recall (R), and F_1 at k=10 (top) and k=100 (bottom) on the DBpedia-Entity, WikIR, and NFCorpus benchmarks.

IR-Model	DBpedia-Entity @10			WikIR @10			NFCorpus @10		
	P	R	F_1	P	R	F_1	P	R	F_1
BM25	0.0759	0.4149	0.1181	0.2101	0.2000	0.1872	0.1929	0.1238	0.1043
VSM-TF	0.0712	0.3962	0.1122	0.1545	0.1387	0.1318	0.1867	0.1288	0.1020
TF-Sum	0.0665	0.3544	0.1034	0.1384	0.1191	0.1147	0.1752	0.1194	0.0955
Boolean OR	0.0741	0.3751	0.1116	0.0848	0.0906	0.0812	0.1495	0.1139	0.0873
Boolean AND	0.0165	0.0821	0.0248	0.0828	0.0872	0.0786	0.1149	0.0848	0.0652
$\mathcal{FDM}$	0.0724	0.4187	0.1145	0.1788	0.1683	0.1579	0.2025	0.1353	0.1087

IR-Model	DBpedia-Entity @100			WikIR @100			NFCorpus @100		
	P	R	F_1	P	R	F_1	P	R	F_1
BM25	0.0119	0.5944	0.0230	0.0553	0.4205	0.0870	0.0528	0.2110	0.0625
VSM-TF	0.0139	0.6502	0.0267	0.0423	0.3006	0.0658	0.0554	0.2221	0.0646
TF- Sum	0.0136	0.6322	0.0262	0.0410	0.2892	0.0635	0.0541	0.2178	0.0630
Boolean OR	0.0129	0.5750	0.0247	0.0358	0.3076	0.0600	0.0488	0.1944	0.0557
Boolean AND	0.0022	0.1106	0.0043	0.0335	0.2807	0.0560	0.0302	0.1099	0.0295
$\mathcal{FDM}$	0.0135	0.6670	0.0260	0.0500	0.3829	0.0806	0.0584	0.2327	0.0677

To assess the relevance of disjunctive attribute implications, we associate each implication with quantitative measures capturing its degree of satisfaction and empirical support in a fuzzy context. Three scores are considered: graded validity, violation-based, and support–confidence.

Assume that L is equipped with a fuzzy implication "$\hookrightarrow$" For a disjunctive attribute implication $A \rightarrowtail B$, we define its graded validity degree by:

$$\mathrm{val}(A \rightarrowtail B) = \bigwedge_{o\in\mathcal{O}} \left(A^{\downarrow^{\blacklozenge}}(o) \hookrightarrow B^{\downarrow^{\blacklozenge}}(o)\right) \quad (15)$$

For each object $o \in \mathcal{O}$, the implication degree $A^{\downarrow^{\blacklozenge}}(o) \hookrightarrow B^{\downarrow^{\blacklozenge}}(o)$ is evaluated, and these values are then aggregated conjunctively over all objects in $\mathcal{O}$. In particular, $\mathrm{val}(A \rightarrowtail B) = 1$ corresponds to full validity as defined in Definition 4. Values close to 1 indicate a highly reliable implication, while lower values correspond to weaker disjunctive dependencies.

We define, for each $o \in \mathcal{O}$, the per-object violation magnitude as

$$\delta_{A,B}(o) = \max\{0,\ A^{\downarrow^{\blacklozenge}}(o) - B^{\downarrow^{\blacklozenge}}(o)\}. \quad (16)$$

This violation can then be aggregated either pessimistically (worst case) or on average:

$$V_{\max}(A \rightarrowtail B) = \max_{o\in\mathcal{O}} \delta_{A,B}(o), \quad (17)$$

$$V_{\mathrm{avg}}(A \rightarrowtail B) = \frac{1}{|\mathcal{O}|}\sum_{o\in\mathcal{O}} \delta_{A,B}(o). \quad (18)$$

The violation score quantifies how strongly an implication is contradicted by the data by measuring, for each object, the excess of the antecedent satisfaction over that of the consequent. Accordingly, $V_{\max}$ captures the worst case, whereas V_{avg} reflects the average over all objects.

The fuzzy support and confidence measures are defined below for the antecedent (A) and the implication ($A \rightarrowtail B$). The support reflects the average degree to which the antecedent (or the implication) holds across the objects of the context, while the confidence evaluates the strength of the disjunctive implication relative to its antecedent.

$$\sup(A) = \frac{1}{|\mathcal{O}|}\sum_{o\in\mathcal{O}} A^{\downarrow^{\blacklozenge}}(o) \quad (19)$$

$$\sup(A \rightarrowtail B) = \frac{1}{|\mathcal{O}|}\sum_{o\in\mathcal{O}} \max\left(A^{\downarrow^{\blacklozenge}}(o),\ B^{\downarrow^{\blacklozenge}}(o)\right) \quad (20)$$

The corresponding fuzzy confidence is defined as

$$\mathrm{conf}(A \rightarrowtail B) = \frac{\sup(A \rightarrowtail B)}{\sup(A)},\ \text{if } \sup(A) > 0 \quad (21)$$

It may be remarked that $A = \emptyset$ makes the implication $A \rightarrowtail B$ useless in practice.

5.3 Adaptive and nonlinear control systems

Built upon fuzzy relational analysis, modalities, and concept-lattice theory, the proposed $\mathcal{FDM}$ approach provides a framework for adaptive and nonlinear control in dynamic environments requiring continuous adjustment. Existing adaptive, fuzzy, and nonlinear synchronization methods [42, 43] rely on multi-mode control structures whose roles vary with operating conditions. Similar coordination mechanisms arise in nonlinear optimal control of industrial processes [44]. In this context, $\mathcal{FDM}$-concept lattices provide a systematic way to organize operating regimes and guide control-mode selection. These lattices are induced by an underlying L-context $\widetilde{\mathcal{K}} = (L, \mathcal{O}, \mathcal{P}, \mathcal{I})$, where $\mathcal{O}$ denotes a set of operating regimes, $\mathcal{P}$ a set of mutually compatible controller properties (attributes), and $\mathcal{I} : \mathcal{O} \times \mathcal{P} \to [0, 1]$ is a fuzzy relation that assigns to each (operating regime, property) pair a degree of admissibility.

In this setting, the $\mathcal{FM}$-operators introduced in 11 and 12 establish a correspondence between operating regimes and controller properties. For any property set P, $P^{\downarrow^{\blacklozenge}}$ determines the fuzzy operating regime in which P is applicable, while $\widetilde{O}^{\uparrow^{\square}}$ identifies the controller properties admissible within a fuzzy operating regime $\widetilde{O}$. Consequently, each $\mathcal{FDM}$-concept $\langle \widetilde{O}, P\rangle$ represents a fuzzy operating regime together with its admissible control components.

To illustrate this construction, we consider a simplified example inspired by the projective lag-synchronization framework proposed in [43]. Let $\mathcal{O} = \{o_1, o_2, o_3\}$ denote the operating regimes corresponding to near synchronization, moderate desynchronization, and strong chaotic behavior, and $\mathcal{P} = \{p_1, p_2, p_3\}$ the nominal tracking, adaptive fuzzy, and robust sliding-mode control components. Their qualitative admissibility[3] is given in Table 9.

Table 9: Example of Fuzzy incidence degrees $\mathcal{I}$ between operating regimes and controller components.

$\mathcal{I}$	p_1	p_2	p_3
o_1	0.9	0.6	0.3
o_2	0.5	0.8	0.7
o_3	0.2	0.6	0.9

Table 10 lists the induced $\mathcal{FDM}$-concepts, and Fig. 3 shows the corresponding lattice, which hierarchically orders operating regimes from specific (bottom) to more conservative (top). Acting as a supervisory layer, the lattice supports mode selection by linking the identified operating regime to the corresponding controller components.

From the perspective of lag-synchronization [43], lower $\mathcal{FDM}$-concepts correspond to configurations in which a specific set of controller properties P determines, through $P^{\downarrow^{\blacklozenge}}$, selectively admissible operating regimes $\widetilde{O}$. These

[3] An imprecise indicator of the suitability of a controller property for a given operating regime, reflecting expert knowledge or observed behavior.

Table 10: Induced $\mathcal{FDM}$-concepts associated with the lag-synchronization L-context.

$\mathcal{FDM}$-Concept	fuzzy operating regime	controller components
$C_\perp$	$\{o_1^{0.0},\ o_2^{0.0},\ o_3^{0.0}\}$	$\emptyset$
C_1	$\{o_1^{0.9},\ o_2^{0.5},\ o_3^{0.2}\}$	$\{p_1\}$
C_2	$\{o_1^{0.6},\ o_2^{0.8},\ o_3^{0.6}\}$	$\{p_2\}$
C_3	$\{o_1^{0.3},\ o_2^{0.7},\ o_3^{0.9}\}$	$\{p_3\}$
C_4	$\{o_1^{0.9},\ o_2^{0.8},\ o_3^{0.6}\}$	$\{p_1, p_2\}$
C_5	$\{o_1^{0.9},\ o_2^{0.7},\ o_3^{0.9}\}$	$\{p_1, p_3\}$
C_6	$\{o_1^{0.6},\ o_2^{0.8},\ o_3^{0.9}\}$	$\{p_2, p_3\}$
$C_\top$	$\{o_1^{0.9},\ o_2^{0.8},\ o_3^{0.9}\}$	$\{p_1, p_2, p_3\}$

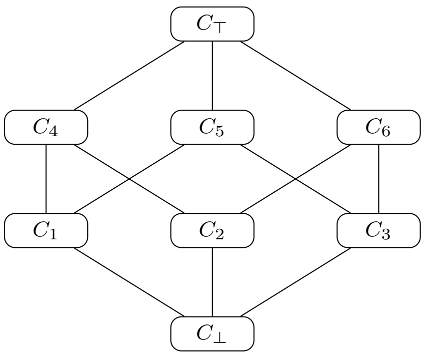

Figure 3: Hierarchical lattice of the $\mathcal{FDM}$-concepts listed in Table 10.

concepts capture control situations in which a single component plays a dominant role. Higher concepts arise when larger sets of properties are considered; the associated operating regimes obtained via $P^{\downarrow^{\blacklozenge}}$ become broader, while the corresponding property sets recovered through $\widetilde{O}^{\uparrow^{\square}}$ reflect more inclusive and conservative control configurations. The top concept $C_\top$ represents maximal compatibility, where the full set of controller properties forms a fixed point of the operators $(.)^{\downarrow^{\blacklozenge}}$ and $(.)^{\uparrow^{\square}}$.

Beyond its static interpretation, the proposed structure naturally admits incremental updates when new operating regimes are observed. The integration of a newly detected regime into the underlying L-context requires only local adjustments of the affected $\mathcal{FDM}$-concepts and their order relations; this can be implemented through minor adaptations of Algorithms 2–4, so that only the impacted concepts and corresponding edges in the Hasse diagram are updated. This mechanism allows the supervisory lattice to evolve consistently with the system dynamics while maintaining interpretability and scalability.

This framework complements classical control laws such as Lyapunov-based adaptive updates, sliding-mode schemes, or optimal feedback policies [42, 43, 44] by providing an interpretable structural layer for decision-making under uncertainty.

6 Discussion

The theoretical developments and experimental results establish that the proposed $\mathcal{FDM}$ framework constitutes a substantial extension of fuzzy FCA through the introduction of a dual-modal semantics. Unlike the one-sided fuzzy FCA approach [23, 4], which is restricted to sufficiency-based reasoning, the $\mathcal{FDM}$ approach integrates necessity and possibility modal operators, yielding a richer semantic characterization of fuzzy concepts.

From a computational and structural standpoint, the results in Tables 5 and 6 make evident the structural and performance differences among the three approaches. Fuzzy FCA exhibits a rapid combinatorial growth in the number of concepts as the number of attributes increases, resulting in substantial execution times. Crisp modal FCA partially controls this growth under fixed thresholds; however, the discretization step used to transform fuzzy contexts into crisp equivalents substantially expands the attribute set and enlarges the resulting lattices, ultimately degrading computational efficiency. In contrast, the $\mathcal{FDM}$ approach maintains a considerably lower number of induced concepts and achieves substantially reduced computation times. This improvement is directly reflected in the structural properties of the induced lattices on real-world datasets. Compared to fuzzy FCA, $\mathcal{FDM}$ induces Hasse diagrams of significantly reduced size and connectivity, with moderated height and width, evidencing a more compact and better-structured lattice organization. Overall, the results show that the $\mathcal{FDM}$ approach effectively limits the combinatorial growth observed in fuzzy and discretization-based modal FCA, yielding structurally manageable lattices and improved practical scalability.

A potential drawback of the proposed approach lies in constructing the Hasse diagram of the $\mathcal{FDM}$-concept lattice, which remains cubic in the worst case, as is the case for other FCA approaches.. While this complexity is acceptable for moderately sized lattices, it may become prohibitive when the number of concepts increases significantly. Consequently, improving the efficiency of lattice construction procedures remains an important direction for future work. In particular, exploiting the incremental and recursive structure of the proposed algorithms may further enhance scalability. Complementary interactive visualization tools could also facilitate the exploration and analysis of large lattices.

Beyond its theoretical and algorithmic contributions, the $\mathcal{FDM}$ framework was empirically evaluated in the context of information retrieval, and its practical relevance was further illustrated in knowledge discovery and adaptive and nonlinear control. In information retrieval, fuzzy dual-modal semantics supports disjunctive query modeling by interpreting $\mathcal{FDM}$-concepts as *answer–query* pairs, while lattice neighbourhood relations enable query refinement.

The $\mathcal{FDM}$-based query model was assessed on standard IR benchmarks and compared with classical lexical retrieval methods. The results, reported in Table 8, show competitive F_1 performance and consistently high recall, particularly when considering larger result sets. In knowledge discovery, the framework enables the derivation of disjunctive attribute implications directly within fuzzy contexts, without requiring complemented contexts, and supports graded validation and filtering. In adaptive and nonlinear control, $\mathcal{FDM}$-lattices provide an interpretable supervisory structure for organizing operating regimes and supporting mode-selection decisions under uncertainty.

7 Conclusion and future works

This work proposes an extension of fuzzy FCA based on fuzzy modal operators ($\mathcal{FM}$-operators) and the corresponding fuzzy dual-modal compositions ($\mathcal{FDM}$-compositions). The algebraic properties of the $\mathcal{FM}$-operators are established, and a fundamental theorem proves that the induced $\mathcal{FDM}$-concepts form a complete lattice. The resulting framework supports graded relational information and necessity–possibility reasoning while yielding concept lattices of controlled size. Through the introduction of $\mathcal{FM}$-operators, concept extraction is extended beyond purely sufficiency-based reasoning.

From a computational standpoint, dedicated algorithms, including an incremental algorithm, were developed for computing $\mathcal{FDM}$-concepts, and their computational complexity was analyzed. Although the construction of the associated Hasse diagram remains cubic in the worst case, experimental results indicate a substantial reduction in the number of induced concepts compared to fuzzy FCA. The applicability of the framework is illustrated in domains such as information retrieval, adaptive and nonlinear control systems, and disjunctive attribute implications.

In this work, we have focused on the dual compositions of $(.)^{\uparrow^{\square}}$ and $(.)^{\downarrow^{\blacklozenge}}$. Future research may investigate alternative compositions of $\mathcal{FM}$-operators and analyze their semantic interpretations. Additional perspectives include the treatment of incomplete fuzzy formal contexts with missing values, the derivation of minimal rule bases for disjunctive attribute implications, and the development of scalable algorithms for large-scale $\mathcal{FDM}$-lattice construction. Further exploration of connections with non-classical logics and emerging domains such as explainable machine learning also constitutes promising research directions.

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