On the Crossing Number of Almost Planar Graphs

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If G is a plane graph and $x, y \in V(G)$, then the dual distance of x and y is equal to the minimum number of crossings of G with a closed curve in the plane joining x and y. Riskin [7] proved that if G_0 is a 3-connected cubic planar graph, and x, y are its vertices at dual distance d, then the crossing number of the graph $G_0 + xy$ is equal to d. Riskin asked if his result holds for arbitrary 3-connected planar graphs. In this paper it is proved that this is not the case (not even for every 5-connected planar graph G_0).

Povzetek: Analizirana je Riskinova teza o planarnih grafih.

1 Introduction

Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. Besides its mathematical interest, there are numerous applications, most notably those in VLSI design [1, 2, 3] and in combinatorial geometry [9]. We refer to [4, 8] and to [10] for more details about such applications.

A *drawing* of a graph G is a representation of G in the Euclidean plane $\mathbb{R}^2$ where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their endvertices. A drawing is *clean* if the interior of every arc representing an edge contains no points representing the vertices of G. If interiors of two arcs intersect or if an arc contains a vertex of G in its interior we speak about crossings of the drawing. More precisely, a *crossing* of $\mathcal{D}$ is a pair $(\{e, f\}, p)$, where e and f are distinct edges and $p \in \mathbb{R}^2$ is a point that belongs to interiors of both arcs representing e and f in $\mathcal{D}$. If the drawing is not clean, then the arc of an edge e may contain in its interior a point $p \in \mathbb{R}^2$ that represents a vertex v of G. In such a case, the pair $(\{v, e\}, p)$ is also referred to as a *crossing* of $\mathcal{D}$.

The number of crossings of $\mathcal{D}$ is denoted by $\mathrm{cr}(\mathcal{D})$ and is called the crossing number of the drawing $\mathcal{D}$. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum $\mathrm{cr}(\mathcal{D})$ taken over all clean drawings $\mathcal{D}$ of G.

A clean drawing $\mathcal{D}$ with $\mathrm{cr}(\mathcal{D}) = 0$ is also called an *embedding* of G. By a *plane graph* we refer to a planar graph together with an embedding in the Euclidean plane. We shall identify a plane graph with its image in the plane.

A nonplanar graph G is *almost planar* if it contains an edge e such that $G - e$ is planar. Such an edge e is called a *planarizing edge*. It is easy to see that almost planar graphs can have arbitrarily large crossing number. In the sequel, we will consider almost planar graphs with a fixed planarizing edge $e = xy$, and will denote by $G_0 = G - e$ the corresponding planar subgraph. By a *plane graph* we mean a planar graph together with its embedding in the plane.

Let G_0 be a plane graph and let x, y be two of its vertices. A simple (polygonal) arc $\gamma : [0, 1] \to \mathbb{R}^2$ is an (x, y)-*arc* if $\gamma(0) = x$ and $\gamma(1) = y$. If $\gamma(t)$ is not a vertex of G_0 for every t, $0 < t < 1$, then we say that γ is *clean*. For an (x, y)-arc γ we define the crossing number of γ with G_0 as

$$\mathrm{cr}(\gamma, G_0) = |\{t \mid \gamma(t) \in G_0 \text{ and } 0 < t < 1\}|.$$

Using this notation, we define the *dual distance*

$$d^*(x, y) = \min\{\mathrm{cr}(\gamma, G_0) \mid \gamma \text{ is a clean } (x, y)\text{-arc}\}$$

and the *facial distance* between x and y,

$$d'(x, y) = \min\{\mathrm{cr}(\gamma, G_0) \mid \gamma \text{ is an } (x, y)\text{-arc}\}.$$

Clearly, $d'(x, y) \le d^*(x, y)$.

Let $G^*_{x,y}$ be the geometric dual graph of the plane graph $G_0 - x - y$. Then $d^*(x, y)$ is equal to the distance in $G^*_{x,y}$ between the two vertices corresponding to the faces of $G_0 - x - y$ containing x and y. This shows that $d^*(x, y)$ can be computed in linear time. Similarly, one can compute $d'(x, y)$ in linear time by using the vertex-face incidence graph (see [6]).

Proposition 1.1. *If G_0 is a planar graph and $x, y \in V(G_0)$, then for every embedding of G_0 in the plane, we have $\mathrm{cr}(G_0 + xy) \le d^*(x, y)$.*

Proposition 1.1 is clear from the definition of d^*. It shows that it is of interest to determine the minimum

$d^*(x, y)$ taken over all embeddings of G_0 in the plane. We refer to [5] for more details and some further extensions.

Riskin [7] proved the following strengthening of Proposition 1.1 in a special case when G_0 is 3-connected and cubic:

Theorem 1.2. *If G_0 is a 3-connected cubic planar graph, then*

$$\mathtt{cr}(G_0 + xy) = d'(x, y).$$

Let us observe that $d'(x, y) = d^*(x, y)$ if G_0 is a cubic graph.

Riskin asked in [7] if Theorem 1.2 holds for arbitrary 3-connected planar graphs. In this paper we show that this is not the case (not even for every 5-connected planar graph G_0).

2 Strange examples

In this section we provide a negative answer to the aforementioned question of Riskin [7] who asked if it is true that for every 3-connected plane graph G_0 and any two of its vertices x, y, the crossing number of $G_0 + xy$ equals $d^*(x, y)$.

Theorem 2.1. *For every integer k, there exists a 5-connected planar graph G_0 and two vertices $x, y \in V(G_0)$ such that $\mathtt{cr}(G_0 + xy) \le 11$ and $d^*(x, y) \ge k$.*

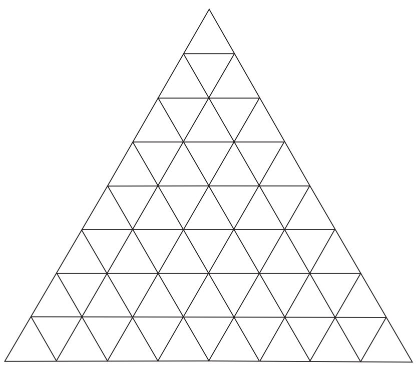

Figure 1: Part of the triangular lattice with side length 8

Proof. Let H_k be the planar graph that is obtained from the icosahedron by replacing all of its triangles, except one, with the dissection of the equilateral triangle with side of length k into equilateral triangles with sides of unit length (as shown in Figure 1 for $k = 8$). This graph is a near triangulation, all its faces are triangles, except one, whose length is $3k$. We may assume that this is the outer face in a plane embedding of H_k. Its boundary is composed of three paths A, B, C of length k joining the original vertices a', b', c' of the icosahedron we started with. Now we add three new vertices, a, b, c and join a with all vertices on A, b with B, and c with C. This gives rise to a 5-connected near triangulation G_k whose outer face is the 6-gon $aa'bb'cc'$. Let us take 5 copies of the graph G_k and let $a_i, a'_i, b_i, b'_i, c_i, c'_i$ be copies of the corresponding vertices on the outer face of the ith copy of G_k, $i = 1, \dots, 5$. Let Q_k be the planar graph obtained from these copies by cyclically identifying b_i with a_{i+1}, adding edges $b'_i c'_{i+1}$ ($i = 1, \dots, 5$, indices modulo 5), and adding two vertices x and y such that x is joined to $a_1, \dots, a_5$ and y is joined to $c_1, \dots, c_5$. See Figure 2. The obtained graph Q_k is planar and it is not difficult to verify that it is 5-connected.

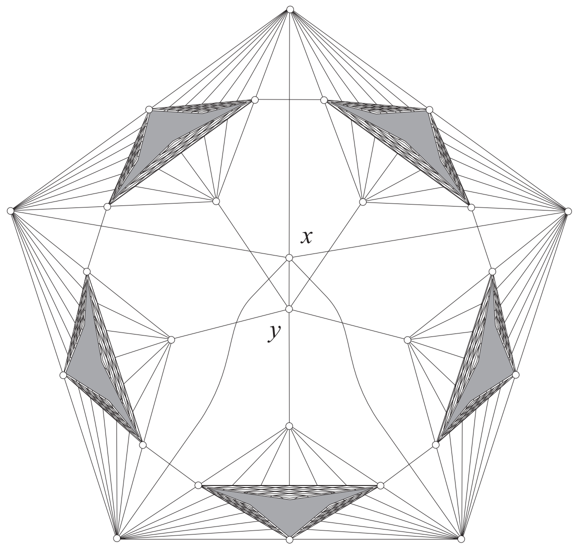

Figure 2: The graph Q_k

It is easy to see that $d^*(x, y) = k + 2$ in Q_k. By putting the vertex x close to y, so that we can draw the edge xy without introducing crossings with other edges, and then redrawing the edges from x to its neighbors as shown in Figure 2, a drawing of $Q_k + xy$ is obtained whose crossing number is 11. □

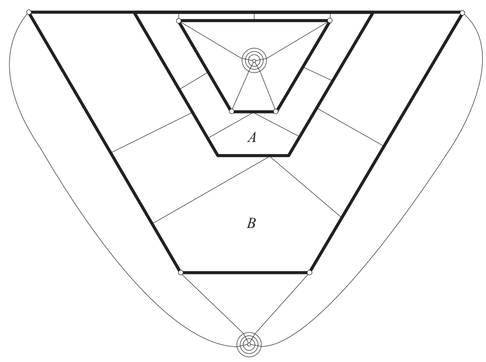

Figure 3: A planar graph for which two flips are needed

The construction of Theorem 2.1 can be generalized such that a similar redrawing as made above for x is necessary also for y (in order to bring these two vertices close together). Such an example is shown in Figure 3, where x

and y are vertices in the centers of the small circular grids on the picture, and where the bold lines represent a "thick" barrier similar to the one used in the graph Q_k in Figure 2. In Figure 4, an optimum drawing of $G_0 + xy$ is shown, where the edge xy is represented by the broken line. In this drawing, neighborhoods of x and y, are redrawn inside the faces denoted by A and B (respectively) in Figure 3.

At the first sight the redrawing described in the above example seems like the worst possibility which may happen – to "flip" a part of the graph containing x and to "flip" a part containing y. If this would be the only possibility of making the crossing number smaller than the one coming from the planar drawing of G_0, this would most likely give rise to a polynomial time algorithm for computing the crossing number of graphs that are just one edge away from a 3-connected planar graph.

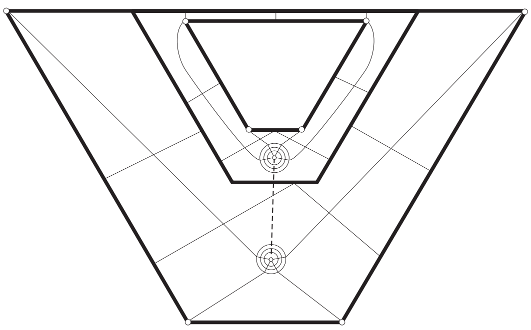

Figure 4: An optimum drawing of $G_0 + xy$

Unfortunately, some more complicated examples show that there are other ways for shortcutting the dual distance from x to y. (Such an example was produced in a discussion with Thomas Böhme and Neil Robertson whose help is greatly acknowledged.) Despite such examples, the following question may still have a positive answer:

Problem 2.2. *Is there a polynomial time algorithm which would determine the crossing number of $G_0 + xy$ if G_0 is planar.*

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