

On the Inverse Problem for Generalized One-Sided Concept Lattices

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Generalized one-sided concept lattices represent a generalization of the classical concept lattices convenient for analysis of object-attribute models with different types of attributes. Formally, to each object-attribute model (represented by the notion of formal context) there is assigned a pair of concept-forming operators. Fixed points of these operators form a hierarchical structure consisting of extent-intent pairs. From the algebraic point of view this structure forms a complete lattice, called the generalized one-sided concept lattice. In this paper we deal with the inverse problem for generalized one-sided concept lattices. For a given generalized one-sided concept lattice we describe an algorithm for finding the corresponding formal context.

Povzetek: Predstavljen je algoritem za preslikavo enostranske mreže konceptov v pripadajoči formalni koncept.

1 Introduction

In mathematics, physics, computer science or engineering there are pairs of problems which are inverses of one another. As examples from mathematics we can mention the multiplication of integers and as the corresponding inverse problem the factorization of a given integer, differentiation and integration of real valued functions or Fourier transform and inverse Fourier transform. From physics we can mention scattering problem, which is to determine how radiation or particles are scattered based on the characteristics of some object (scatterer) and inverse scattering problem of determining characteristics of an object based on data of how it scatters incoming radiation or particles.

At first glance, the meaning of the term ‘inverse problem’ seems obvious. It is the problem which is associated to some other (direct) problem, one that presumably preceded the inverse problem and which has been studied extensively for some time and is better known. Our inverse problem concerns determination of characteristics of object-attribute models in fuzzy modification of Formal Concept Analysis (FCA), so called generalized one-sided concept lattices.

The theory of concept lattice, also called Formal Concept Analysis is a theory of data analysis for identification of conceptual structures among data sets. As an effective tool for data analysis, Formal Concept Analysis has been extensively applied to a variety of fields such as data min-

ing, decision making, information retrieval, machine learning and knowledge discovery. The main notion of this theory is the notion of a formal context, represented by a binary relation between the set of objects and the set of attributes, specifying which objects have what attributes. From a formal context, one can construct object-attribute pairs known as the formal concept. The family of all formal concepts forms an algebraic structure called the concept lattice, which reflects the relationship of generalization and specialization among particular concepts. The reader can find an extensive account of the mathematical foundations of FCA in [7].

In many real applications, however, the relationship may be many-valued (fuzzy). Therefore, some attempts have recently been devoted to introduce fuzzy concept lattice with properties similar to the classical ones. We mention approaches [2, 3] based on residuated lattices or multi-adjoint concept lattices [11]. A very important class of fuzzy concept lattices is formed by the one-sided concept lattices, where usually objects are considered as crisp subsets and attributes obtain fuzzy values, cf. [9] or [10]. In this case interpretation of object clusters is straightforward as in classical FCA. Consequently, all known applications developed for classical concept lattices can be used in the theory of one-sided concept lattices. Recently there was a generalization of all one-sided approaches (the so-called generalized one-sided concept lattices, see [6] for more details), which allows one to consider different types of structure for

truth degrees (represented by complete lattices). From this point of view it is applicable to a very wide spectrum of the real object-attribute models where methods of the classical FCA are appropriate, cf. [1, 5, 4, 8, 14, 15].

As we have already mentioned, our aim is to deal with the inverse problem for generalized one-sided concept lattices. The paper is organized as follows: in the next section we give a brief overview of the notions concerning generalized one-sided concept lattices. We recall some algebraic notions like Galois connections, complete lattices or closure systems. Our main result, i.e., an algorithm for the inverse problem is presented in Section 3. In particular we will deal with the decision problem, i.e., whether a given collection of pairs forms a generalized one-sided concept lattice, and consequently we describe a method for determining the formal context (object-attribute model) corresponding to a given generalized one-sided concept lattice.

2 Generalized one-sided concept lattices

In this section we describe a fuzzy generalization of classical concept lattices, the so-called generalized one-sided concept lattices, cf. [6] and [13].

The main idea of fuzzifications of classical FCA is the usage of graded truth. The structure L of truth degrees forms a so-called complete lattice, i.e., it is partially ordered, contains the smallest and the greatest element (representing the values **false** and **true**, respectively), moreover, for any subset $H \subseteq L$ there exists $\bigvee H$ (the least upper bound or supremum) and $\bigwedge H$ (the greatest lower bound or infimum). In classical logic, each proposition is either true or false, hence classical logic is bivalent. It is common to represent the classical logic truth value structure as a two-element chain, i.e., the two-element set $\{0, 1\}$ with $0 < 1$. In this case the value 0 represents false and 1 represents true. In fuzzy logic, to each proposition there is assigned a truth degree from some richer scale L of truth degrees. If to the propositions Φ and Ψ are assigned truth degrees $\|\Phi\| = a$ and $\|\Psi\| = b$, then $a \leq b$ means that Φ is considered less true than Ψ . In object-attribute models the typical propositions are of the form “object has attribute in degree a ”. The well-known examples of truth structures used in various modifications of fuzzy logic are: the real unit interval $[0, 1]$, Boolean algebras, MV algebras, or, more generally, residuated lattices.

The set of all L -fuzzy sets over some universe U is defined as the set of all functions

$$f : U \rightarrow L,$$

denoted by symbol L^U . In order to define generalized one-sided concept lattices we will use the notion of direct product. If L_i for $i \in I$ is a family of lattices the *direct product* $\prod_{i \in I} L_i$ is defined as the set of all functions

$$f : I \rightarrow \bigcup_{i \in I} L_i$$

such that $f(i) \in L_i$ for all $i \in I$ with the “componentwise” order, i.e., $f \leq g$ if $f(i) \leq g(i)$ for all $i \in I$. If $L_i = L$ for all $i \in I$ we get the direct power L^I . In this case the direct power L^I represents the structure of L -fuzzy sets, hence the direct product of lattices can be seen as a generalization of the notion of L -fuzzy sets. The direct product of lattices forms a complete lattice if and only if all members of the family are complete lattices. Straightforward computations show that the lattice operations in the direct product $\prod_{i \in I} L_i$ of complete lattices are calculated componentwise, i.e., for any subset $\{f_j : j \in J\} \subseteq \prod_{i \in I} L_i$ we obtain

$$\begin{aligned} (\bigvee_{j \in J} f_j)(i) &= \bigvee_{j \in J} f_j(i), \\ (\bigwedge_{j \in J} f_j)(i) &= \bigwedge_{j \in J} f_j(i), \end{aligned}$$

where these equalities hold for each index $i \in I$.

In order to introduce the notion of generalized one-sided concept lattices as a generalization of FCA we will assume only one minimal condition, i.e., that the structures of truth degrees form complete lattices.

In the mathematical theory of fuzzy concept lattices, the main role is played by special pairs of mappings between complete lattices, commonly known as Galois connections. Hence, we provide necessary details regarding Galois connections and related topics.

Let (L, \leq) and (M, \leq) be complete lattices and let $\varphi : L \rightarrow M$ and $\psi : M \rightarrow L$ be maps between these lattices. Such a pair (φ, ψ) of mappings is called a *Galois connection* if the following condition is fulfilled:

$$p \leq \psi(q) \quad \text{if and only if} \quad \varphi(p) \geq q.$$

Galois connections between complete lattices are closely related to the notion of closure operator and closure system. Let L be a complete lattice. By a *closure operator* in L we understand a mapping $c : L \rightarrow L$ satisfying:

- (a) $x \leq c(x)$ for all $x \in L$,
- (b) $c(x_1) \leq c(x_2)$ for $x_1 \leq x_2$,
- (c) $c(c(x)) = c(x)$ for all $x \in L$ (i.e., c is idempotent).

A subset X of the complete lattice L is called a *closure system* in L if X is closed under arbitrary meets. We note that this condition guarantees that (X, \leq) is a complete lattice, in which the infima are the same as in L , but the suprema in X may not coincide with those from L . For a closure operator c in L , the set $\text{FP}(c)$ of all fixed points of c (i.e., $\text{FP}(c) = \{x \in L : c(x) = x\}$) is a closure system in L . Conversely, for a closure system X in L , the mapping $C_X : L \rightarrow L$ defined by $C_X(x) = \bigwedge \{u \in X : x \leq u\}$ is a closure operator in L . Moreover these correspondences are inverses of each other, i.e., $\text{FP}(C_X) = X$ for each closure system X in L and $C_{\text{FP}(c)} = c$ for each closure operator c in L .

Next we describe the mathematical framework for the generalized one-sided concept lattices. We start with the

definition of formal context, from which there is defined a pair of mappings forming a Galois connection.

A 4-tuple (B, A, L, R) is said to be a *generalized one-sided formal context* if B is a non-empty set of objects, A is a non-empty set of attributes, $L: A \rightarrow \text{CL}$ is a mapping from the set of attributes to the class of all complete lattices. In this case $L(a)$ represents a particular structure of truth value degrees for each attribute $a \in A$. Finally, $R: B \times A \rightarrow \bigcup_{a \in A} L(a)$ with $R(b, a) \in L(a)$ is an incidence relation, which represents a degree from the structure $L(a)$ in which an element $b \in B$ has a given attribute $a \in A$.

The power set (set of all subsets) of a set B will be denoted by $\mathbf{P}(B)$. Let (B, A, L, R) be a generalized one-sided formal context. Then there is defined a pair of mappings $\perp: \mathbf{P}(B) \rightarrow \prod_{a \in A} L(a)$ and $\top: \prod_{a \in A} L(a) \rightarrow \mathbf{P}(B)$ as follows:

$$X^\perp(a) = \bigwedge_{b \in X} R(b, a), \quad (1)$$

$$g^\top = \{b \in B : \forall a \in A, g(a) \leq R(b, a)\}. \quad (2)$$

The pair (\perp, \top) forms a Galois connection between $\mathbf{P}(B)$ and L^A . The composition of mappings \perp and \top forms a closure operator in $\mathbf{P}(B)$ and similarly the composition of \top and \perp forms a closure operator in $\prod_{a \in A} L(a)$. Hence, subsets of the form $X^{\perp\top}$ for any $X \subseteq B$ are closed subsets with respect to the closure operator defined above. As it is known, the closed subsets of any closure operator form a complete lattice with respect to the inherited partial order from the underlying complete lattice structure (in this case $\mathbf{P}(B)$). This fact stands behind the formal definition and characterization of concept lattices.

For a given generalized one-sided formal context (B, A, L, R) the symbol $\mathfrak{C}(B, A, L, R)$ will denote the set of all pairs (X, g) with $X \subseteq B, g \in \prod_{a \in A} L(a)$, satisfying

$$X^\perp = g \quad \text{and} \quad X = g^\top.$$

In this case, the set X is usually referred to as the *extent* and g as the *intent* of the concept (X, g) . Further we define a partial order on $\mathfrak{C}(B, A, L, R)$ as follows:

$$(X_1, g_1) \leq (X_2, g_2) \text{ iff } X_1 \subseteq X_2 \text{ iff } g_1 \geq g_2.$$

Let (B, A, L, R) be a generalized one-sided formal context. The set $\mathfrak{C}(B, A, L, R)$ with the partial order defined above forms a complete lattice, where

$$\bigwedge_{i \in I} (X_i, g_i) = \left(\bigcap_{i \in I} X_i, \left(\bigvee_{i \in I} g_i \right)^{\top\perp} \right)$$

$$\bigvee_{i \in I} (X_i, g_i) = \left(\left(\bigcup_{i \in I} X_i \right)^{\perp\top}, \bigwedge_{i \in I} g_i \right)$$

for each family $(X_i, g_i)_{i \in I}$ of elements from $\mathfrak{C}(B, A, L, R)$.

The lattice $\mathfrak{C}(B, A, L, R)$ is called the *generalized one-sided concept lattice*.

R	a_1	a_2	a_3	a_4
b_1	2	0.25	0	1
b_2	3	0.50	1	0
b_3	1	0.35	0	0
b_4	1	0.25	0	1
b_5	2	0.70	1	0

Table 1: Incidence relation R .

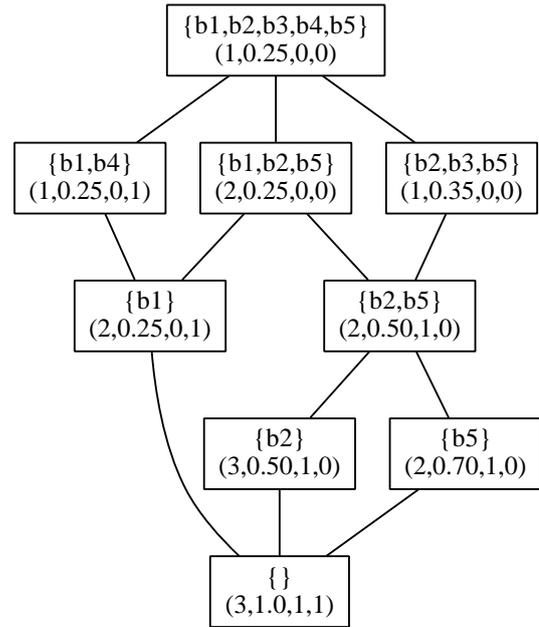


Figure 1: Generalized one-sided concept lattice corresponding to (B, A, L, R) .

We provide a small example of a generalized one-sided formal context and the corresponding generalized one-sided concept lattice. Consider the five-element set of objects $B = \{b_1, b_2, b_3, b_4, b_5\}$, and the four-element set of attributes $A = \{a_1, a_2, a_3, a_4\}$ with $L(a_1) = 4, L(a_2) = [0, 1]$ and $L(a_3) = L(a_4) = 2$. In this case 4 denotes the four-element chain $0 < 1 < 2 < 3$ and 2 denotes the two-element chain $0 < 1$. Finally, the incidence relation R is given in Table 1. Obviously the triple (B, A, L, R) forms a generalized one-sided formal context. The corresponding generalized one-sided concept lattice is depicted in Figure 1.

3 The inverse problem for generalized one-sided concept lattices

After introducing the necessary theoretical background for the direct problem (creation of a generalized one-sided concept lattices from a given formal context), we can provide

the precise definition of our inverse problem. Let $B \neq \emptyset$ be a set of objects, $A \neq \emptyset$ be a set of attributes, $L(a)$ be a system of complete lattices (truth structures under consideration) and \mathcal{C} be a set consisting of some pairs (X, g) where $X \subseteq B$ and $g \in \prod_{a \in A} L(a)$. Decide whether there exists (in affirmative case also find) an incidence relation $R: B \times A \rightarrow \bigcup_{a \in A} L(a)$ such that $\mathcal{C} = \mathcal{C}(B, A, L, R)$.

In order to decide the inverse problem for generalized one-sided concept lattices we will use the following well-known characterization of Galois connections involving dual isomorphism of closure systems, cf. [12]:

Let L, M be complete lattices. Any Galois connection between L and M is fully determined by dually isomorphic closure systems in L and M .

In order to provide more details, suppose that X_1 and X_2 are closure systems in L, M respectively, and $f: X_1 \rightarrow X_2$ is a dual isomorphism between complete lattices (X_1, \leq) and (X_2, \leq) . Then a pair $(c_{X_1} \circ f, c_{X_2} \circ f^{-1})$, where c_{X_1}, c_{X_2} are the closure operators corresponding to X_1 and to X_2 , forms a Galois connection between L and M .

Given \mathcal{C} as input, we want to decide if there exists a Galois connection (φ, ψ) between $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ such that

$$\mathcal{C} = \{(X, g) : X = \psi(g), g = \varphi(X)\}.$$

If this condition is satisfied, then one can find a corresponding formal context such that $\mathcal{C} = \mathcal{C}(B, A, L, R)$, hence this condition is equivalent to our decision problem. In order to solve this issue, we will use the previous result concerning Galois connections, i.e., we verify that the projections of \mathcal{C} form dually isomorphic closure systems.

The first step is to decide whether the sets

$$C_1 = \{X \subseteq B : (\exists g)(X, g) \in \mathcal{C}\}$$

and

$$C_2 = \{g \in \prod_{a \in A} L(a) : (\exists X)(X, g) \in \mathcal{C}\}$$

form closure systems in $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$, respectively.

Hence we must check if C_1 is closed under arbitrary intersections and C_2 is closed under arbitrary meets.

The second step is to decide whether C_1 and C_2 form dually isomorphic closure systems. We recall that a surjective mapping $f: C_1 \rightarrow C_2$ is a dual isomorphism if for all $X_1, X_2 \in C_1$ it is true that

$$X_1 \subseteq X_2 \quad \text{iff} \quad f(X_1) \geq f(X_2). \quad (3)$$

This condition will be satisfied if for all $X_1, X_2 \in C_1$ it holds that $X_1 \subseteq X_2$ implies $g_1 \geq g_2$ where g_1 and g_2 are such that $(X_1, g_1) \in \mathcal{C}$ and $(X_2, g_2) \in \mathcal{C}$. Similarly, it must hold for all $g_1, g_2 \in C_2$ that $g_1 \geq g_2$ implies $X_1 \subseteq X_2$ where X_1 and X_2 are such that $(X_1, g_1) \in \mathcal{C}$ and $(X_2, g_2) \in \mathcal{C}$.

Now we can summarize the whole procedure in the following algorithm (Algorithm 1).

Algorithm 1 for deciding existence of the incidence relation R

Input: a set of pairs \mathcal{C}

Output: answer YES or NO

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1:  $C_1 \leftarrow \{X : (X, g) \in \mathcal{C}\}$     ▷ Set of first components
2:  $C_2 \leftarrow \{g : (X, g) \in \mathcal{C}\}$     ▷ Set of second components
3: for all  $X_1, X_2 \in C_1$  do
4:   if  $X_1 \cap X_2 \notin C_1$  then ▷  $C_1$  is not a closure system
5:     return NO
6:   end if
7: end for
8: for all  $g_1, g_2 \in C_2$  do
9:   if  $g_1 \wedge g_2 \notin C_2$  then ▷  $C_2$  is not a closure system
10:    return NO
11:  end if
12: end for
13: for all  $X_1, X_2 \in C_1$  such that  $X_1 \subseteq X_2$  do
14:    $g_1 \leftarrow g$  where  $(X_1, g) \in \mathcal{C}$ ,  $g_2 \leftarrow g$  where
       $(X_2, g) \in \mathcal{C}$ 
15:   if  $g_1 \not\geq g_2$  then    ▷  $C_1$  and  $C_2$  are not dually
      isomorphic
16:     return NO
17:   end if
18: end for
19: for all  $g_1, g_2 \in C_2$  such that  $g_1 \geq g_2$  do
20:    $X_1 \leftarrow X$  where  $(X, g_1) \in \mathcal{C}$ ,  $X_2 \leftarrow X$  where
       $(X, g_2) \in \mathcal{C}$ 
21:   if  $X_1 \not\subseteq X_2$  then    ▷  $C_1$  and  $C_2$  are not dually
      isomorphic
22:     return NO
23:   end if
24: end for
25: return YES    ▷  $C_1$  and  $C_2$  are dually isomorphic

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The correctness of this algorithm can be proved using the above-mentioned relationship between Galois connections and dually isomorphic closure systems.

In **for all** loop (line 3 - 7) it is decided whether C_1 forms a closure system in $\mathbf{P}(B)$. Similarly, **for all** loop (line 8 - 12) decides whether C_2 forms a closure system in the direct product of lattices $\prod_{a \in A} L(a)$. If C_1 and C_2 form closure systems, then the next step is to decide whether the correspondence $f: X \mapsto g, (X, g) \in \mathcal{C}$ is a dual isomorphism between C_1 and C_2 . This is verified in the two **for all** loops (line 13 - 18, line 19 - 24). Let us note that the condition (3) guarantees that the correspondence f is injective. If $f(X_1) = f(X_2)$ then $f(X_1) \geq f(X_2)$ and $f(X_2) \geq f(X_1)$ which yields $X_1 \subseteq X_2$ and $X_2 \subseteq X_1$. Since the inclusion relation is antisymmetric, we obtain $X_1 = X_2$. Moreover, we deal with finite structures only, hence f is surjective too, and consequently it is a bijection.

The algorithm returns the affirmative answer if and only if C_1 and C_2 are dually isomorphic closure systems. In this case, there is a Galois connection between $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ corresponding to the input set \mathcal{C} .

Let \mathcal{C} be an input set and $n = |\mathcal{C}|$ denote the number of

all pairs in \mathfrak{C} . Obviously $|C_1| \leq n$ and $|C_2| \leq n$. Since there are $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ different two-element subsets of an n -element set, we obtain that two **for all** loops (line 3 - 7, line 8 - 12) have no more than $c \cdot \frac{n \cdot (n-1)}{2} \in O(n^2)$ repetitions. Here we assume that the verification whether $X_1 \cap X_2 \in C_1$ and $g_1 \wedge g_2 \in C_2$ can be done in constant time $c \in \mathbb{R}$. Similarly, the time complexity of other two **for all** loops is $O(n^2)$, hence Algorithm 1 is in $O(n^2)$ time complexity class according to the size of the input set n .

In what follows we will deal with the second problem, hence suppose that the decision problem is answered affirmatively. We describe a procedure for finding the incidence relation corresponding to \mathfrak{C} . For this purpose we recall the following assertion concerning Galois connections between power sets and direct products of complete lattices, cf. [6].

Let (ϕ, ψ) be a Galois connection between $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$. Then there exists a generalized one-sided formal context (B, A, L, R) such that $\phi(X) = X^\perp$ for all $X \subseteq B$ and $\psi(g) = g^\top$ for all $g \in \prod_{a \in A} L(a)$.

According to the definition (1) of the mapping $\perp: \mathbf{P}(B) \rightarrow \prod_{a \in A} L(a)$ we obtain for all $b \in B$ and all $a \in A$

$$\{b\}^\perp(a) = \bigwedge_{b' \in \{b\}} R(b', a).$$

Since the right side of this equality expresses the infimum over the one-element set $\{R(b, a)\}$ we obtain

$$\{b\}^\perp(a) = \bigwedge_{b' \in \{b\}} R(b', a) = R(b, a).$$

This yields that the value of the incidence relation $R(b, a)$ is fully determined by the a -th projection of $\{b\}^\perp$. Moreover, due to [6] the following assertion is valid:

Let B be a non-empty set and $L(a)$ be a system of complete lattices. Then any two Galois connections $(\phi_1, \psi_1), (\phi_2, \psi_2)$ between $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ are equal if and only if

$$\phi_1(\{b\})(a) = \phi_2(\{b\})(a)$$

for all $b \in B$ and for all $a \in A$.

Since we assume that particular projections of the elements of \mathfrak{C} form dually isomorphic closure systems, we already know that there is a Galois connection (ϕ, ψ) between $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ such that the corresponding fixed points of (ϕ, ψ) form the lattice \mathfrak{C} . Hence we define the incidence relation $R: B \times A \rightarrow \bigcup_{a \in A} L(a)$ as follows:

$$R(b, a) := \phi(\{b\})(a), \text{ for all } b \in B, a \in A.$$

From this definition it follows that

$$\phi(\{b\})(a) = R(b, a) = \{b\}^\perp(a)$$

for all $b \in B, a \in A$ and, due to the above-mentioned assertion, this yields $\perp = \phi$ and $\top = \psi$. In order to determine all the values $R(b, a)$ we must find the corresponding values of $\phi(\{b\})(a)$ in the generalized one-sided concept lattice \mathfrak{C} .

For this purpose we use the characterization of the one part of Galois connections $\phi: \mathbf{P}(B) \rightarrow \prod_{a \in A} L(a)$ as the composition of the closure operator c on the set B and the dual isomorphism f between closure systems in $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ respectively. In this case $\phi = c \circ f$, i.e., $\phi(X) = f(c(X))$ for all $X \subseteq B$.

The isomorphism f is given directly by the ordered pairs in the generalized one-sided concept lattice \mathfrak{C} . If $(X, g) \in \mathfrak{C}$, then the dual isomorphism f is given by

$$f(X) = g, \text{ for all } X \subseteq B.$$

Hence the main goal is to determine the values $c(\{b\})$ for all $b \in B$.

From the definition of closure operator, it follows that for each $b \in B$ the value $c(\{b\})$ is the smallest subset appearing in \mathfrak{C} which contains the given element b . For this reason it is convenient to deal with minimal elements of the concept lattice \mathfrak{C} . If (X, g) is a minimal element of \mathfrak{C} then for all $b \in X$ it holds $c(\{b\}) = X$ and consequently

$$\phi(\{b\}) = f(c(\{b\})) = f(X) = g$$

for all $b \in X$. In the next step, we can remove this concept from the concept lattice \mathfrak{C} and find another minimal element, say (X_1, g_1) . Let us note that after removing any of the concepts (X, g) from \mathfrak{C} the resulting structure is no longer a lattice in general. However it is still a partially ordered set, thus the notion of a minimal element can be used again. In this case $c(\{b\}) = X_1$ for all $b \in X_1 \setminus X$. In this way we can proceed, until we exhaust all the elements in the object set B .

The whole procedure is described in more detail in Algorithm 2.

The correctness of this algorithm follows from the fact that for an object b and an attribute a the value $R(b, a)$ is uniquely determined by the value $\phi(\{b\})(a)$ where ϕ represents one part of the Galois connection (ϕ, ψ) corresponding to the input set \mathfrak{C} . Moreover $\phi(\{b\}) = \phi(X)$ where X is the closure of the element b in Galois connection (ϕ, ψ) , i.e., $X = \psi(\phi(\{b\}))$. Consequently $\phi(\{b\}) = \phi(X) = g$ with $(X, g) \in \mathfrak{C}$. Closures of the one-element subsets are minimal with respect to the closure operator, hence in the **while** loop (line 2 - 13) the algorithm works with the minimal concepts in \mathfrak{C} . In the **for all** loop (line 5 - 12) the values $R(b, a)$ for $b \in X \setminus S, a \in A$ are determined (**for all** loop (line 7 - 9)). Let $b \in B$ be an object, $(X, g) \in \mathfrak{C}$ be a minimal concept such that $b \in X$ and suppose that $b \notin S$, i.e., the values $R(b, a)$ for $a \in A$ are not determined yet. Then X is unique with this property and for all $a \in A$ the value $R(b, a)$ is determined correctly. By contrary assume that there is another subset X' with $(X', g') \in \mathfrak{C}, b \in X'$ and $X \not\subseteq X'$. Then $b \in X \cap X' \subsetneq X$ and $(X \cap X', g'') \in \mathfrak{C}$ for some $g'' \in \prod_{a \in A} L(a)$ since the first components of \mathfrak{C} form a closure system in $\mathbf{P}(B)$. This yields a contradiction to the fact that X is the minimal concept in \mathfrak{C} for which $R(b, a)$ is not determined.

Finally, we describe the time complexity of Algorithm 2. Let \mathfrak{C} be its input. Again, denote by n the number of

Algorithm 2 for finding the incidence relation R **Input:** a generalized one-sided concept lattice \mathcal{C} **Output:** the incidence relation R

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1:  $S \leftarrow \emptyset$   $\triangleright$  Set of objects  $b$  for which  $R(b, a)$  has
   already been determined
2: while  $S \neq B$  do  $\triangleright$  Repeat until all values are
   determined
3:    $m \leftarrow (X, g) : (X, g)$  a minimal element in  $\mathcal{C}$   $\triangleright$ 
   Find a minimal concept
4:    $\mathcal{C} \leftarrow \mathcal{C} \setminus \{m\}$   $\triangleright$  Remove the minimal concept  $m$ 
   from  $\mathcal{C}$ 
5:   for all  $b \in X$  where  $(X, g) = m$  do
6:     if  $b \notin S$  then  $\triangleright b$  has no value  $R(b, a)$  yet
7:       for all  $a \in A$  do
8:          $R(b, a) \leftarrow g(a)$   $\triangleright$  Determination of the
   value  $R(b, a)$ 
9:       end for
10:       $S \leftarrow S \cup \{b\}$   $\triangleright$  Add the object  $b$  to the set
    $S$ 
11:     end if
12:   end for
13: end while
14: return  $R$ 

```

concepts in \mathcal{C} and denote by k the number of all objects in the set B . In the worst case the **while** loop (line 2 - 13) has n repetitions (this happens when \mathcal{C} is a chain). A minimal concept of \mathcal{C} can be found in $O(|\mathcal{C}|)$ time (see Algorithm 3).

Algorithm 3 for finding a minimal concept in \mathcal{C} **Input:** a generalized one-sided concept lattice \mathcal{C} **Output:** a minimal concept $m = (X, g)$

```

1:  $m \leftarrow$  an arbitrary element in  $\mathcal{C}$ 
2: for all  $m' \in \mathcal{C}$  do
3:   if  $m' < m$  then
4:      $m \leftarrow m'$ 
5:   end if
6: end for
7: return  $m$ 

```

Since $X \subseteq B$ for all concepts $(X, g) \in \mathcal{C}$, the **for all** loop (line 5 - 12) has at most k repetitions. Other loops can be executed in constant time, hence we obtain that the time complexity of Algorithm 2 is

$$\sum_{i=0}^{n-1} k \cdot c \cdot (n - i) = ck \cdot \frac{n \cdot (n + 1)}{2} \in k \cdot O(n^2).$$

Since in many real situations $|B| \leq |\mathcal{C}|$, we can conclude that the time complexity of Algorithm 2 is $O(n^3)$.

4 Conclusion

In this paper we have presented an algorithm for the inverse problem of generalized one-sided concept lattices, i.e., how

to determine a generalized one-sided formal context from a given generalized one-sided concept lattice. This provides a possibility to express information about object-attribute models with different types of attributes in the form of hierarchical structures represented by generalized one-sided concept lattices.

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