

Variational Iteration Method for Solving Electrocardiography Inverse Problem

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In this paper, the variational iteration method is proposed as a solution to an inverse problem in electrocardiography. The aim is to obtain approximate solutions to the model with the lowest error rate. This is crucial in determining the patient's heart activity and facilitating rapid medical intervention. The main challenges in accurately computing clinically relevant maps of cardiac depolarization stem from the ill-posed nature of the continuous problem and the presence of noise in the data. To tackle these difficulties, we have developed regularizing iterative algorithms based on domain decomposition techniques. These algorithms reformulate the inverse problem into several cases, depending on the solution area. This formulation has enabled us to establish a new stopping criterion that is more responsive and accurately reflects the behavior of the error on the non-accessible part of the boundary. The numerical results obtained through the variational iteration method demonstrate that the proposed approach effectively captures the error behavior on the non-accessible boundary. An additional advantage of these approaches is their ability to reduce execution time through their parallelized versions. Thus, we have successfully demonstrated the effectiveness of these methods in terms of quality (accuracy of approximation) and quantitative aspects (computation cost).

Povzetek: Članek predstavlja variacijsko iteracijsko metodo za reševanje inverznih problemov v elektrokardiografiji z nizko stopnjo napake.

1 Introduction

Due to the importance of mathematical simulation in science, several mathematical models were built for the electrical activity in the heart, including the bidomain model [1,2,3]. The heart muscle is anisotropic, the effects of these properties on heart tissue are discussed using the bidomain electrophoresis model, it is a microscopic model that describes the bioelectric behavior of the heart [4], The mathematical model of electrophysiology of the heart is a dynamic system that quantitatively describes the electrical processes occurring in the tissues of the heart [5]. The bidomain model is a system of nonlinear and convergent ordinary differential equations (ODEs), which used to model the gastric electrophysiology [6]. The use of the bidomain model is necessary for the correct modeling of the defibrillation response, and the error arising from the numerical solution of bidomain models is relatively small; various types of boundary condition can be imposed that can take into account the leakage of current to the surrounding tissues other than the myocardium [7]. In order for the model results to be useful, it is necessary to obtain accurate input of the models [8,9], Therefore, we will try to be precise in

choosing the initial conditions to get better results and by using the Variational Iteration Method (VIM) which described and used to give approximate solutions for some non-linear problems [10,11,12], This method relying on The initial condition and based on Taylor series approximation, which applied as center difference, front difference, and back difference schemes[13], This method has the ability to reduce the volume of calculations and easily overcome the difficulty of the perturbation method[14]; Later, this method was used based on Laplace transformations to get the solution in the form of a series [15], there is no need to linearize or treat the nonlinear terms to apply the method on nonlinear differential equations [16].

2 Bidoman model

It is a mathematical way to describe the electrical activity of heart which developed since the invention of the ECG measuring by Einthoren [17]. It is used to simulate cardiac electrical tissue computationally and to study the stimulation of cardiac tissue and defibrillation of the heart. The heart domain denoted by Ω and the conductivity tensor of intercellular and extracellular by

σ_i, σ_e , let v_m be atransmembrane potential and u_i , u_e be electric potential of intercellular and extracellular respectively, such that $v_m = u_i - u_e$. The bidomain model in two dimensional defined as:

$$\begin{cases} A_m \left(C_m \frac{\partial v_m}{\partial t} + I_{ion}(v_m, w) \right) - \text{div}(\sigma_i \nabla v_m) = \text{div}(\sigma_i \nabla u_e) & \text{in } \Omega \times]0, T[\\ \text{div}((\sigma_i + \sigma_e) \nabla u_e) = -\text{div}(\sigma_i \nabla v_m) & \text{in } \Omega \times]0, T[\\ \partial_t w + g(v_m, w) = 0 & \text{in } \Omega \times]0, T[\\ \sigma_i \nabla v_m \cdot n = -\sigma_i \nabla u_e \cdot n & \text{on } \Sigma \times]0, T[\\ (\sigma_i + \sigma_e) \nabla u_e \cdot n = -\sigma_i \nabla v_m \cdot n & \text{on } \Sigma \times]0, T[\end{cases} \quad (1)$$

Where A_m is defined to be the surface to volume ratio of the cell membrane, C_m the membrane capacitance, I_{ion} is the current due to the ion exchange. $g(v_m, w)$ is a function of fields having the same dimension of w which is the concentrations of different chemical, n is the outward unit normal on Ω , and $\Sigma = \Gamma_{endo} \cup \Gamma_{epi}$, Γ_{endo} internal boundaries in the rated endocardia and Γ_{epi} an outer boundaries, epiciedial [18].

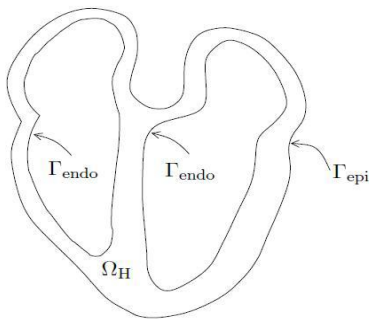


Figure 1: Cardiac field distribution

3 Variational iteration method

The variational iteration method (VIM) proposed to find a numerical solution for linear and nonlinear equations by (He ,1998,1999) [10], then it was developed to solve various types of integral equations and ordinary, partial and fractional differential equations.

consider the general equation

$$L u + N u = g(x) \quad (2)$$

where L is linear operator, N is the nonlinear operator and g is inhomogeneous term. Assuming $u_0(x)$ is the initial solution, then the variational iteration technique becomes:

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda(\xi) (L u_n + N u_n - g) dx \quad (3)$$

With u_0 as initial condition, we can rewrite this equation as follow

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) (L u_n(\xi) + N u_n(\xi) - g(\xi)) d\xi \quad (4)$$

Where λ is the general Lagrange multiplier [15]

$$\lambda(\xi) = \frac{(-1)^m}{(m-1)!} (\xi - x)^{m-1} \quad (5)$$

m the highest order of the derivative, the final solution is $u(x) = \lim_{n \rightarrow \infty} u_n(x)$.

4 Application

The system (1) will be resolved using VIM in two cases:

Case (1): The system (1) can be solved using VIM when $\lambda = 1$ in $\Omega \times]0, T[$

$$\begin{cases} v_m^{n+1} = v_m^n + \lambda \int_0^t (A_m (C_m \frac{\partial v_m^n}{\partial t} + I_{ion}(v_m^n, w_m^n)) - \text{div}(\sigma_i \nabla v_m^n) - \text{div}(\sigma_i \nabla u_e^n)) dt \\ u_e^{n+1} = u_e^n + (u_e^n(t, 0, y) + \frac{\partial(u_e^n(t, 0, y))}{\partial x} x) - \frac{1}{(\sigma_i + \sigma_e)} (\sigma_i v_m^n - \sigma_i v_m^n(t, 0, y) - \sigma_i \frac{\partial v_m^n(t, 0, y)}{\partial x} x) \\ + \frac{\lambda}{(\sigma_i + \sigma_e)} \int_0^x \int_0^x (-\sigma_i \frac{\partial^2 v_m^n}{\partial y^2} - (\sigma_i + \sigma_e) \frac{\partial^2 u_e^n}{\partial y^2}) dx dx \\ w_m^{n+1} = w_m^n + \lambda \int_0^t (\frac{\partial w_m^n}{\partial t} + g(v_m^n, w_m^n)) dt \end{cases} \quad (6)$$

with the initial conditions [18]

$$v_m^0(0, x, y) = v_0(x), w_m^0(0, x, y) = w_0(x) \quad (7)$$

$$I_{ion}(v_m, w) = \frac{W}{T_{in}} V^2 (V - 1) - \frac{V}{T_{out}},$$

$$g(v_m, w) = \begin{cases} \frac{w-1}{T_{open}} & \text{if } v \leq v_{gate} \\ \frac{w}{T_{closed}} & \text{if } v > v_{gate} \end{cases} \quad (8)$$

Where $T_{in} \leq T_{out} \leq T_{open}, T_{closed}$ and $0 < v_{gate} < 1$, are given positive constants.

Using iterative formula (6) with the initial conditions:

$$\begin{cases} Z \\ u_e^0 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - x^2y^3 - 3x^4 - 6x^2y^2 - 3y^4 + 3x^2 + 3y^2 - 1 \\ u_i^0 = 2x^6 + 6x^4y^2 + 6x^2y^4 + 2y^6 - 2x^2y^3 - 6x^4 - 12x^2y^2 - 6y^4 + 6x^2 + 6y^2 - 2 \\ v_m^0 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - x^2y^3 - 3x^4 - 6x^2y^2 - 3y^4 + 3x^2 + 3y^2 - 1 \\ w_m^0 = \frac{1}{(v_m^0)^2 (v_m^0 - 1)} \end{cases} \quad (9)$$

Then the first iterative formula

$$\begin{cases} u_e^1 = h + \frac{rt}{s} \\ u_i^1 = 2h + 2\frac{rt}{s} \\ v_m^1 = h + \frac{rt}{s} \\ w_m^1 = \frac{1}{h + (\frac{rt}{s})^2 (h - 1 + \frac{rt}{s})} \end{cases} \quad (10)$$

where

$$\begin{cases} r = x^6 T_{in} + 3x^4 y^2 T_{in} + 3x^2 y^4 T_{in} + y^6 T_{in} - x^2 y^3 T_{in} \\ - 3x^4 T_{in} - 6x^2 y^2 T_{in} - 3y^4 T_{in} + 3x^2 T_{in} + 3y^2 T_{in} + T_{out} T_{in} - T_{in} - T_{out} \\ s = T_{in} C_m T_{out} \\ h = x^6 + 3x^4 y^2 + 3x^2 y^4 + y^6 - x^2 y^3 - 3x^4 - 6x^2 y^2 - 3y^4 + 3x^2 + 3y^2 - 1 \\ j = 26.4t - 13.2x^2 y t + 79.2x^4 t + 79.2y^4 t + 158.4x^2 y^2 t - 105.6y^2 t - 4.4y^3 t - 105.6x^2 t \end{cases}$$

and at the boundary conditions

$$\begin{cases} [2h \quad 2h - 12(2y^2 + 6x^2 + \frac{1}{3}y^3 - y^4 - 6x^2y^2 - 5x^4 - 1)t \quad \dots] \\ [2h \quad 2h - 12(6y^2 + 2x^2 + x^2y - 5y^4 - 6x^2y^2 - x^4 - 1)t \quad \dots] \\ [h/2 \quad (h/2) + (-66x^4 - 79.2x^2y^2 - 13.2y^4 + 4.4y^3 + 79.2x^2 + 26.4y^2 - 13.2)t \quad \dots] \\ [h/2 \quad (h/2) + (-13.2x^4 - 79.2x^2y^2 - 66y^4 + 13.2x^2y + 26.4x^2 + 79.2y^2 - 13.2)t \quad \dots] \end{cases}$$

The general form

$$\begin{cases} V_{VIM} = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - x^2y^3 - 3x^4 - 6x^2y^2 - 3y^4 + 3x^2 + 3y^2 - 1 + h + \frac{rt}{s} + \dots \\ u_{VIM} = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - x^2y^3 - 3x^4 - 6x^2y^2 - 3y^4 + 3x^2 + 3y^2 - 1 + h + j + \dots \\ w_{VIM} = \frac{1}{h^2+(h-1)} + \frac{1}{h+(\frac{rt}{s})^2(h-1+\frac{rt}{s})} + \dots \end{cases} \quad (11)$$

The exact solution for system (1) with initial conditions (9) and the values $T_{open} = 1$; $T_{closed} = 1$; $T_{in} = 1$; $T_{out} = 1$; $C_m = 1$; $\sigma_i = 1$; $\sigma_e = 0.2$ is $V_m = ((x^2 + y^2 - 1)^3 - x^2y^3)e^t$ (12)

Table 1: Demonstrates comparison absolute error of numerical solution using VIM with the exact solution when $n = 3, x \in [-1.5, 1.5]; y \in [-1.5, 1.5]; t = 0.25$.

V_{exact}	V_{VIM}	$Errore_{VIM}$
55.77659477	55.77296875	3.626020×10^{-3}
19.00253201	19.00459321	2.061200×10^{-3}
4.944311443	4.948546790	4.235347×10^{-3}
0.9159855499	0.9208438900	4.858340×10^{-3}
0.1857372348	0.1907085100	4.971275×10^{-3}
0.1036097736	0.09859375000	5.016024×10^{-3}
0.6066703134	0.6015764900	5.093823×10^{-3}
1.040166975	1.035006110	5.160865×10^{-3}
1.040189078	1.035028210	5.160868×10^{-3}
0.6120414440	0.6069467900	5.094654×10^{-3}

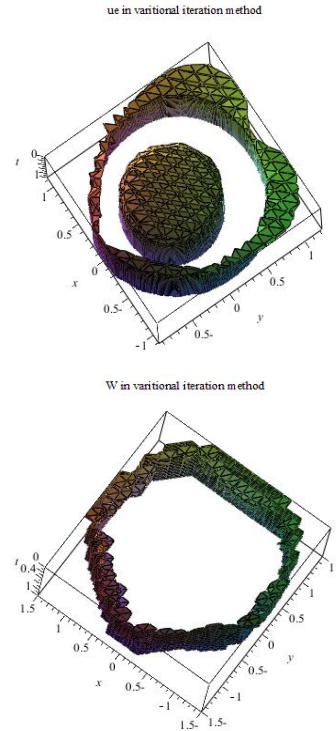


Figure 2: Solution path for system (1) using VIM

We notice from the table (1) and figure (2), that the numerical solution approaches the exact solution based on the parameters found using system (11), where the absolute error rate reaches 10^{-3} with few iterations $n=3$, Whereas the interval that is approved is $[-1.5, 1.5] \times [-1.5, 1.5]$.

Case 2: If we Segmentation the area of heart into an internal and an external area only, after substituting the second equation into the first and third equations of the system (1), we get the following system in $\Omega \times]0, T[$:

$$\begin{cases} A_m \left(C_m \frac{\partial v_m}{\partial t} + I_{ion}(v_m, w) \right) + \text{div}(\sigma_e \nabla u_e) = 0 & \text{in } \Omega \times]0, T[\\ \frac{\partial w}{\partial t} + g(v_m, w) = 0 & \text{in } \Omega \times]0, T[\end{cases} \quad (13)$$

and on the boundaries of the heart in figure (1), the system (1) will be as:

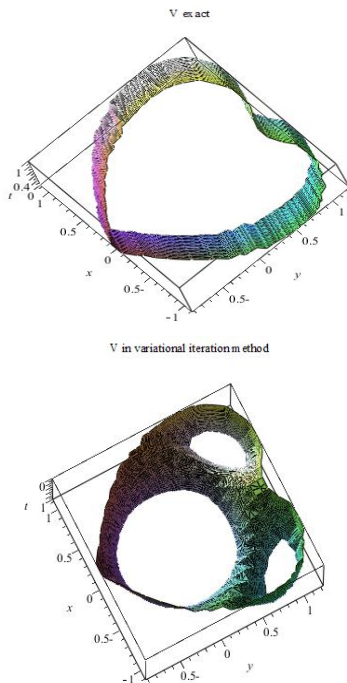
$$\begin{cases} A_m \left(C_m \frac{\partial v_m}{\partial t} + I_{ion}(v_m, w) \right) = 0 & \text{on } \Sigma \times]0, T[\\ \frac{\partial w}{\partial t} + g(v_m, w) = 0 & \text{on } \Sigma \times]0, T[\end{cases} \quad (14)$$

The systems (13) and (14) can be solved via VIM with the initial conditions (7) and:

$$I_{ion}(v_m, w) = g(v_m, w) = v_m - w \quad (15)$$

Then

$$\begin{cases} v_m^{n+1} = v_m^n + \lambda \int_0^t \left(A_m \left(C_m \frac{\partial v_m^n}{\partial t} + I_{ion}(v_m^n, w_m^n) \right) - \text{div}(\sigma_e \nabla u_e^n) \right) dt & \text{in } \Omega \times]0, T[\\ w_m^{n+1} = w_m^n + \lambda \int_0^t \left(\frac{\partial w_m^n}{\partial t} + g(v_m^n, w_m^n) \right) dt & \text{in } \Omega \times]0, T[\end{cases} \quad (16)$$



At the boundary conditions, the system (14) becomes:

$$\begin{cases} v_m^{n+1} = v_m^n + \lambda \int_0^t (A_m (C_m \frac{\partial v_m^n}{\partial t} + I_{ion}(v_m^n, w_m^n))) dt & \text{in } \Sigma \times]0, T[\\ w_m^{n+1} = w_m^n + \lambda \int_0^t (\frac{\partial w_m^n}{\partial t} + g(v_m^n, w_m^n)) dt & \text{in } \Sigma \times]0, T[\end{cases} \quad (17)$$

The exact solution for system (1) with initial conditions (9) and the values $T_{open} = 1$; $T_{closed} = 1$; $T_{in} = 1$; $T_{out} = 1$; $C_m = 1$; $\sigma_i = 1$; $\sigma_e = 0$ is $V_m = ((x^2 + y^2 - 1)^3 - x^2 y^3) e^t$ (18)

At the boundaries of the heart, the boundary conditions towards x are used when $x = x_0$ with $y = y_0$, $y = y_L$ in system (17), while for the boundaries of the heart towards y , then the boundary conditions towards y are used when $y = y_0$ with $x = x_0$, $x = x_L$ in system (17).

Table 2: Demonstrates comparison absolute error of numerical solution using VIM with the exact solution when $n = 3, x \in [-1.5, 1.5]; y \in [-1.5, 1.5]; t = 0.25$.

V_{exact}	V_{VIM}	$Errore_{VIM}$
45.66601344	45.66601758	3.941900×10^{-4}
15.55795734	15.55795875	1.343000×10^{-4}
4.048059831	4.048060198	3.494200×10^{-5}
0.7499455391	0.7499456071	6.473700×10^{-6}
0.1520687861	0.1520688000	1.312600×10^{-6}
0.0848285079	0.0848285156	7.322400×10^{-7}
0.4966996426	0.4966996875	4.287600×10^{-6}
0.8516166907	0.8516167679	7.351300×10^{-6}
0.8516347875	0.8516348647	7.351300×10^{-6}
0.5010971524	0.5010971978	4.325500×10^{-6}

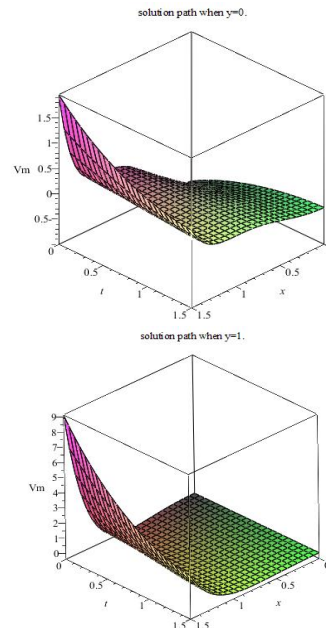
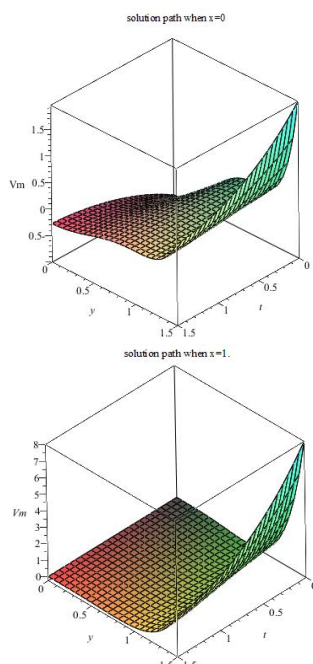


Figure 3: Solutions path for system (1) at the boundary conditions using VIM.

We note from the table (2), the approximation solution is significantly improved when dividing the domain solution area of the heart, bringing the absolute error to 10^{-6} .

5 Conclusion

Using simulation to form a system of differential equations and solve them numerically offers the potential to reduce the error rate in disease diagnosis and avoid excessive treatment dosages. This approach is widely applicable across different fields, including electrocardiogram simulation. The objective is to develop a non-invasive imaging technique that utilizes electrical activation patterns on the surface of the heart to provide meaningful diagnostic information. The variational iteration method provides a solution to inverse Cauchy problems and can be implemented within a general framework. Its utilization is based on robust domain decomposition techniques, which have proven to be effective in various applications. Building on this foundation, we aim to further explore the application of other domain decomposition methods to solve the specific inverse problem at hand.

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