Potential Impact of Climate Change on Groundwater Level Declination in Bangladesh: A Mathematical Modeling Computation

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Groundwater is one of the most important natural resources for the human being due to its ecological diversity. It has become a vastly vital and dependable source of water in all climatic regions together with each urban and rural areas of developed and developing countries like Bangladesh. Groundwater level declines for many reasons, some of which occurs due to natural phenomena and others are caused by human's activities and it has been declining since the introduction of deep tube wells and shallow tube wells after 1970s. Excessive demand of water, evaporation, pollution, deforestation, poor storage, low rainfall, urbanization, frequent pumping of water are the important causes of groundwater level declination in Bangladesh as well as all over the world. Taking these potential effects of climatic change into account, we formulate a mathematical model of groundwater level declination with the help of a system of nonlinear ordinary differential equations (ODEs). The model has been analyzed by finding the existence of equilibrium points and also the conditions of stability and instability near the equilibrium points have been derived by using the stability theory of non-linear differential equations with Lyapunov function and phase portrait analysis. Finally, the numerical simulations have been performed to illustrate the effect of pollution, deforestation, frequent pumping of water and evaporation on the groundwater level in support of analytical findings. Our study shows that, groundwater level decreases significantly due to over pumping, pollution, deforestation and global warming.

Povzetek: Padajoča raven podzemne vode je povezana s človekovimi aktivnostmi in naravnimi pojavi. V študiji je bil uporabljen matematični model za analizo vpliva onesnaževanja, sečnje gozdov in prekomernega črpanja podzemne vode.

1 Introduction

Groundwater is an essential part of public water supply and food production in Bangladesh and also significant natural resource that constitutes 95 percent of the freshwater [27]. About three billion people depend directly upon for drinking water and 40% of the world's food is produced from irrigated agriculture that relies mainly on groundwater. Moreover, in the rural area, farmers are used freshwater about 70% for the purpose of cultivation and by 2050, nine billion people will depend upon an approximated 50% increase in irrigated agricultural production [22]. In Bangladesh, 45,400 deep tubewells, 1,533,920 shallow tubewells and 170,470 low lift pumps are employed presently to provide water for cultivation and using groundwater around 79% of the total cultivated area is irrigated [27]. Nowadays, groundwater acts as the mainstay for agriculture in the southern region of Asia like India, Sri Lanka, Pakistan, Bangladesh, and the Northern China. The water reserved in the ground can be compared to money kept in a bank account. If we pick off money at a swift rate than savings new money, we will ultimately start having account-supply problems. Over pumping of water from the ground level rapidly than it is replenished for the lengthy period causes' similar problems [26].

Thus, groundwater level declination is a serious threat to the environment. It is primarily caused by sustained groundwater pumping. Moreover, due to poor reserving system, lots of rain water is wasted and the scientific development in procuring rain water is very poor. In case of saltwater contamination in the deep within the ground, a vast amount of usable water is reduced. Rainfall has diminished considerably over the years and it is the cause of large-scale deforestation and also rigorous climatic changes. These add to the reduction in water resources as well as groundwater resources. For agricultural use ground water is pumped more frequently, this can be done by fixing a tube well. Most of the industrial waste water are dumped to the water sources. This can be considered another reason for declination of water resources. Due to global warming and change in the climate, huge amounts of surface water and ground water is being evaporated, these also accumulate to the reduction in water resources as well as groundwater resources [51].

The influential consequences of groundwater declination are considered as drying up of tubewells, reduction of water in rivers, lakes i.e. in the land surface, degradation of water quality, increased the costs of pumping due to interrelation between ground and surface water. The rivers, streams and lakes connected to groundwater can also have their supply declined while groundwater is overused [22].

Mathematical biology has become a well-recognized area of environmental research which is the most enjoyable applications of basic sciences [15, 42]. The aim of the mathematical biology is to represent a real phenomenon by mathematics and develop a model on the basis of conservation principle of biological processes, exploitation techniques by using the tools of mathematics. It has both practical and theoretical applications in the field of biotechnological research, biomedicine, ecology and so on [8, 31, 36]. Mathematical simulations and modeling play an incredible role for providing quantitative insight into different field of science. It has already contributed to a better understanding of the mechanisms in various field nowadays. Mathematical modeling has gotten attention because modeling and simulation of any physical phenomena allows us for rapid assessment. So, it is mainly used to describe the real phenomena which lead to design better prediction, management and control strategies [9].

Groundwater declination problem has been discussed by many researchers with different view of points. Declination rate and trend line analysis of groundwater level underneath Dhaka and Gazipur city in Bangladesh is described in [44]. A scientific study is discussed for the management and development of groundwater by using integrated remote sensing with geographic information systems [1]. The quality assessment of groundwater for the usefulness of drinking water is analyzed in [34]. A Mathematical model was proposed and inspected to study the impacts of predicted climate mutation on recharge and groundwater levels [49], whereas the effects of climate change on vegetation is characterized in [5] that is the important factor to store groundwater for long period. Another model based on the causes and quantification was also approached to describe the declination of groundwater level and aquifer dewatering in Dhaka metropolitan area in Bangladesh [230]. However, some articles [2-4, 7, 10-13, 15, 19-20, 21, 24, 29, 32, 38-48, 50, 52, 55-57] are referred for more details in the explanation of global climate change in Bangladesh as well as all over the world, interactions among atmospheric, surface and groundwater with the significance factors for depletion of ground layer and development of mathematical model on groundwater level declination.

In this paper, we would like to propose a new mathematical model on groundwater level declination. The goal of this study is to describe the impacts of climate change in the environment like deforestation, pollution, evaporation and frequent pumping of water on groundwater level. We formulate a mathematical model and study the existence and stability of the model with nonlinear Lyapunov function. Lastly, numerical simulations will be performed to show the effectiveness of the analytical analysis of groundwater model.

The manuscript is organized as follows. In section 2.1, we describe the present scenery of groundwater level declination perspective to Bangladesh; materials and methods of the study are given in section 2.2. We propose a new mathematical model for groundwater level declination in section 3.1. We evaluate the boundedness and positivity of the solution of the model, in subsections 3.2.1 and 3.2.2. and 3.3 we calculated the equilibrium analysis, in 3.4, we discuss the local stability and global stability of the model at both of the equilibrium points. The numerical simulation and sensitivity analysis of the groundwater model is then presented in section 4. In section 5, we observe the dynamical behavior of the model by plotting phase-plane diagram and we conclude the study with some recommendations for preventive steps to reduce the declination of groundwater level in section 6.

2.1 Groundwater declination perspective to bangladesh

Groundwater is an imperishable, renewable and potential natural resource in Bangladesh. In the wet or monsoon season (June–September), groundwater sustains it’s dynamism by recharging, through rainfall and flooding and seceding during the dull season (March–May) due to withdrawal for various application, especially for cultivation. Agricultural yield of Bangladesh has successively increased over the last few decades while we have reasons to be happy about to fact that we must ensure this is not just a fleeting surge but a sustainable propensity. Professor E. M. Bean from University of Guelph observed that the region irrigated by groundwater in Bangladesh increased from 4 to 70 percent from 1972 to 1999 [19]. This research also investigated that due to the dependency of surface water and groundwater to one another, barrages built upstream of rivers entering Bangladesh block down the natural flow rates and it affects the groundwater recharge in the areas of downstream [6]. The average annual rainfall of the country is at least 2000 mm per year in most of the districts in Bangladesh with the exception of the comparatively dry western area of Rajshahi, where the average rainfall is approximated 1650 mm. The northeastern part of Bangladesh receives the greatest average precipitation, sometimes over 4000 mm per year and near about 80% of Bangladesh's rain falls during the monsoon season [56]. Before the extensive evocation of groundwater, water tables in the mostly unconfined aquifers of Bangladesh were generally shallow with a feeble seasonal vacillating propensity from the mid-2000s and onwards. Groundwater level were near or very close to the ground surface in some places after the monsoon period and at the end of the dry season, groundwater tables declined due to evaporation, evapotranspiration and inter-basin flow out of the aquifers. Average depth of groundwater level in Bangladesh is given from 2000 to 2018 in Figure 1.

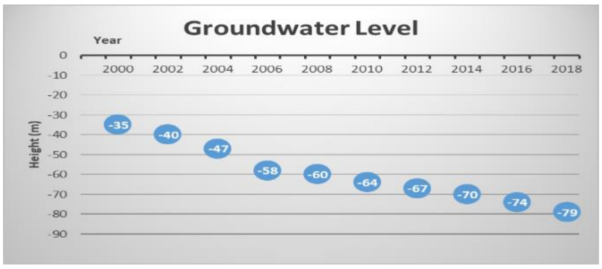

Figure 1: Groundwater level has declined during dry season since 2000 to 2018 in Bangladesh [6].

2.2 Materials and methods

In this study, a mathematical model on groundwater level declination with the help of a system of nonlinear ordinary differential equations has been developed. We evaluate the biological feasibility and mathematical exactness of the model with boundedness and positivity analysis of the solution by using Lemma 1 and Lemma 2. We calculate the equilibrium analysis and study the local stability and global stability of the model at both of the equilibrium points by satisfying Routh-Hurwitz criterion through the sign of the eigen value from the characteristic equation and using Lyapunov stability theorem. Finally, numerical simulations have been performed in MATLAB programming language based on the parametric values collected from secondary data as shown in Table 1 which are estimated from research articles, existing literature, annual reports of government, and non-government institutions. To find the parametric values from collective data, different types of qualitative methods like rough order of magnitude (ROM) or conceptual estimation, system observation, least square method have been performed for securing data authentically.

3 Mathematical formulation of the model

In this section, we introduce the background of the model with some basic assumptions and formulate a mathematical model on groundwater level declination.

3.1 Basic assumption and description of the model

To construct a mathematical model on groundwater level, firstly we have to know the biological interactions and background among the state variables (i.e. atmosphere, surface and ground water). For this purpose, we discuss the water flow management with water cycle in the earth and assume some basic assumption as follows:

According to hydrologic cycle, water vaporizes to become the part of the atmosphere from the oceans and the land surface; then water vapor is deported and lifted in the atmosphere until it condenses and precipitates on the surface or the oceans. Through plantation and vegetation precipitated water can be intercepted for overland flow on the ground surface. Water then infiltrates into the ground as subsurface flow and discharge into streams as surface runoff. Finally, the intercepted water and surface runoff returns again to the atmosphere through evaporation directly from the soil and vegetation surface or transpired from plant leaves. In this way hydrologic cycle continues [28].

This study aims to show the effect of the important parameters used in the model like deforestation, pollution, evaporation and frequent pumping of water on groundwater level declination. The interaction of atmosphere, surface and ground water is shown in the Figure 2.

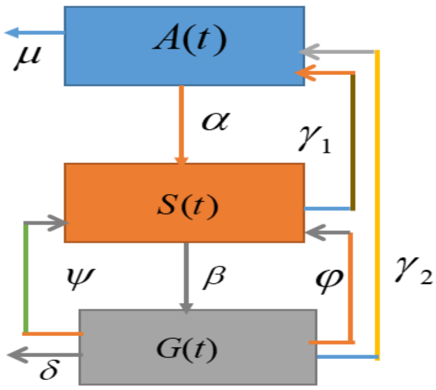

Figure 2: The schematic diagram with biological interactions of groundwater model.

Considering the basic assumptions and schematic diagram in **Figure 2**, we formulate the model with the following system of nonlinear differential equations:

$$\frac{dA}{dt} = \gamma_1 S(t) + \gamma_2 G(t) - \alpha A(t) - \mu A(t) \tag{1}$$

$$\frac{dS}{dt} = \alpha A(t) - \beta G(t)S(t) - \gamma_1 S(t) + \psi G(t)S(t) + \varphi G(t) \tag{2}$$

$$\frac{dG}{dt} = \beta G(t)S(t) - \gamma_2 G(t) - \psi G(t)S(t) - \varphi G(t) - \delta G(t) \tag{3}$$

with initial conditions $A(0) = A_0$, $S(0) = S_0$ and $G(0) = G_0$ and the total amount of water, $N(t) = A(t) + S(t) + G(t)$.

Here, $A(t)$, $S(t)$ and $G(t)$ are the three state variables that represent the levels of atmospheric water, surface water and groundwater respectively at time t.

In the model (1)-(3), we have considered the parameter α as precipitation rate from atmosphere to surface water and μ is the dissipation rate of atmospheric water. The infiltration rate from surface to groundwater be β whereas γ_1 and γ_2 are the evaporation rate from surface and groundwater to atmospheric water respectively. The surface water is polluted at the rate ψ, moreover δ and φ be the deforestation rate and rate of frequent pumping of water from ground level.

3.2 Analysis of the model

We have to analyze the qualitative behavior of the solutions in the neighborhood of the equilibrium points. For the analysis of the model (1)-(3), a closed set has been considered as

$$\Omega = \left\{ (A(t), S(t), G(t)) \in \mathbb{R}_+^3 : 0 \le N(t) \le \frac{\eta}{\mu+\delta} \right\}$$

where, η is a constant.

3.2.1 Boundedness of the model

To prove the system mathematically and biologically well posed, the following lemma has to be satisfied.

Lemma 1: The region $\Omega = \left\{ (A(t), S(t), G(t)) \in \mathbb{R}_+^3 : 0 \le N(t) \le \frac{\eta}{\mu+\delta} \right\}$ is a positively invariant set for the model (1)-(3).

Proof: Since the total amount of water is $N(t)$, then $N(t) = A(t) + S(t) + G(t)$.

The rate of change of total amount of water is

$$\frac{dN}{dt} = \frac{dA}{dt} + \frac{dS}{dt} + \frac{dG}{dt}$$

$$= \gamma_1 S + \gamma_2 G - \alpha A - \mu A + \alpha A - \beta GS - \gamma_1 S + \psi GS + \varphi G$$
$$+ \beta GS - \gamma_2 G - \psi GS - \varphi G - \delta G - \mu G$$
$$= -\mu A - \mu S - \delta G - \mu G$$
$$= -\mu(A + S + G) - \delta G$$
$$= -\mu N - \delta(N - A - S)$$

After simplifying, we get the differential inequality as

$$\frac{dN}{dt} \le \eta - \mu N - \delta N$$

$$\Rightarrow \frac{dN}{dt} + (\mu + \delta) N \le \eta$$

(4)

By solving the differential equation (4), we obtain

$$N(t) \le \frac{\eta}{\mu+\delta} + \left(N_0 - \frac{\eta}{\mu+\delta} \right) e^{-(\mu+\delta)}$$

Taking limit as $t \to \infty$, we get $0 < N(t) \le \frac{\eta}{\mu+\delta}$

(5)

Thus, we conclude that the region Ω is the positively invariant set induced by the model (1)-(3) [33, 36]. Therefore, the model is both mathematically and biologically well-posed in the region Ω.

Hence, the **Lemma 1** is proved.

3.2.2 Positivity of the solution of the model

Since the state equations describes the rate of change of water, so it is required to prove that all the state variables used in the model (1)-(3) are positive.

Lemma 2: If $A(0) > 0, S(0) > 0$ and $G(0) > 0$ then the solutions $A(t), S(t), G(t)$ of the model (1) are all positive [46].

Proof: To prove the Lemma 2, we will use the system of differential equations (1)-(3).

The first equation of the model (1)-(3), can be written as

$$\frac{dA}{dt} \ge -(\alpha + \mu) A$$

$$\Rightarrow \frac{dA}{dt} + (\alpha + \mu) A \ge 0$$

(6)

The integrating factor (I.F) of (6) is given by

$$I.F = e^{\int (\alpha+\mu) dt} = e^{(\alpha+\mu)t}$$

Multiplying the I.F on the both side of (6) and integrating we get

$$e^{(\alpha+\mu)t} \frac{dA}{dt} + (\alpha + \mu) e^{(\alpha+\mu)t} \ge 0$$

$$\Rightarrow A(t) \ge c_1 e^{-(\alpha+\mu)t}$$

(7)

where c_1 is an integrating constant

To find the value of constant c_1 we apply the initial condition at $t = 0, A(0) = A_0$. we get $A_0 \ge c_1$.

Putting the value of c_1 in (7) we obtain

$$A(t) \ge A_0 e^{-(\alpha+\mu)t} .$$

(8)

Since $A_0 > 0$ and very larger than α and μ.

Therefore $A(t) > 0$ for all $t \ge 0$.

Similarly, we obtain from the differential equations (2) and (3) that, $S(t) \ge 0$, and $G(t) \ge 0$ for all $t \ge 0$.

Hence the **Lemma 2** is proved.

3.3 Existence of the equilibrium

To find the equilibrium points of the model (1)-(3), we have to solve

$$\frac{dA}{dt} = \frac{dS}{dt} = \frac{dG}{dt} = 0$$

Then the system takes the following form:

$$\gamma_1 S + \gamma_2 G - \alpha A - \mu A = 0 \tag{9}$$

$$\alpha A - \beta GS - \gamma_1 S + \psi GS + \varphi G = 0 \tag{10}$$

$$\beta GS - \gamma_2 G - \psi GS - \varphi G - \delta G = 0 \tag{11}$$

For pollution free and without pumping from groundwater, we consider $G = G_0$ (i.e. initial value of $G(t)$).

Let $\overline{E}_1\left(\overline{A},\overline{S},\overline{G}\right)$ be the pollution free equilibrium point. Using $G=G_0$ in the system (9)-(11), and solving the system of algebraic equation, we get

$$\overline{A}=\frac{\gamma_1\overline{S}+\gamma_2 G_0}{\alpha+\mu} \quad \text{and} \quad \overline{S}=\frac{\gamma_2+\varphi+\delta}{\beta-\psi} \tag{12}$$

Hence, the pollution free equilibrium point is

$$\overline{E}_1\left(\overline{A},\overline{S},\overline{G}\right)=\left(\frac{\gamma_1 S+\gamma_2 G_0}{\alpha+\mu},\frac{\gamma_2+\varphi+\delta}{\beta-\psi},G_0\right) \tag{13}$$

Again, let $E_2^*\left(A^*,S^*,G^*\right)$ be the equilibrium point of the model (1)-(3) when all the activities are performed in the groundwater (i.e. the general equilibrium point), then we obtain by solving the system (9)-(11) as

$$A^*=\frac{\gamma_1 S^*+\gamma_2 G^*}{\alpha+\mu}, \quad S^*=\frac{\gamma_2+\varphi+\delta}{\beta-\psi} \quad \text{and}$$

$$G^*=\frac{S^*\left[\alpha\gamma_1-\gamma_1\left(\alpha+\mu\right)\right]}{\left(\beta S^*+\psi S^*+\varphi\right)\left(\alpha+\mu\right)-\alpha\gamma_2} \tag{14}$$

Thus, the equilibrium point of the system is

$$E_2^*\left(A^*,S^*,G^*\right)=\left(\begin{array}{c}\dfrac{\gamma_1 S^*+\gamma_2 G^*}{\alpha+\mu},\dfrac{\gamma_2+\varphi+\delta}{\beta-\psi},\\ \dfrac{S^*\left[\alpha\gamma_1-\gamma_1\left(\alpha+\mu\right)\right]}{\left(\beta S^*+\psi S^*+\varphi\right)\left(\alpha+\mu\right)-\alpha\gamma_2}\end{array}\right) \tag{15}$$

where, $A^*=\dfrac{\alpha\delta^2\gamma_1+\alpha\delta\gamma_1\gamma_2+\alpha\delta\gamma_1\varphi}{\alpha(\beta-\psi)\left(\alpha\delta+\delta\mu+\gamma_2\mu\right)}$, $S^*=\dfrac{\delta+\gamma_2+\varphi}{\beta-\psi}$

and $G^*=\dfrac{\delta\gamma_1\mu+\gamma_1\gamma_2\mu+\gamma_1\mu\varphi}{\left(\alpha\delta\psi+\delta\mu\psi+\gamma_2\mu\psi-\alpha\beta\delta-\beta\delta\mu-\beta\gamma_2\mu\right)}$

3.4 Stability analysis

In this section, we have to perform stability analysis at the equilibrium point $\overline{E}_1$ and E_2^*.

3.4.1 Stability at $\overline{E}_1\left(\overline{A},\overline{S},\overline{G}\right)$

Theorem 1: The equilibrium point $\overline{E}_1\left(\overline{A},\overline{S},\overline{G}\right)$ of the model (1)-(3) is asymptotically stable if $a_1>0,\ a_3>0$ and $a_1a_2>a_3$, otherwise it is unstable [33, 53].

Proof: In order to prove Theorem 1, we first find Jacobian matrix of the model (1)-(3) [21].

The Jacobian matrix of the system (1)-(3) is given by

$$J(A,S,G)=\begin{pmatrix}-\alpha-\mu & \gamma_1 & \gamma_2\\ \alpha & -\beta G-\gamma_1+\psi G & -\beta S+\psi S+\varphi\\ 0 & \beta G-\psi G & \beta S-\gamma_2-\psi S-\varphi-\delta\end{pmatrix} \tag{16}$$

At the equilibrium point $\overline{E}_1\left(\overline{A},\overline{S},\overline{G}\right)$, the Jacobian matrix (16) takes the following form

$$J\left(\overline{A},\overline{S},\overline{G}\right)=\begin{pmatrix}-\alpha-\mu & \gamma_1 & \gamma_2\\ \alpha & -\beta G_0-\gamma_1+\psi G_0 & -\beta S+\psi\overline{S}+\varphi\\ 0 & \beta G_0-\psi G_0 & \beta\overline{S}-\gamma_2-\psi\delta-\varphi-\delta\end{pmatrix} \tag{17}$$

The characteristic equation for the eigen value λ is given as

$$\left|J-\lambda I\right|=0$$

$$\begin{vmatrix}-\alpha-\mu-\lambda & \gamma_1 & \gamma_2\\ \alpha & -\beta G_0-\gamma_1+\psi G_0-\lambda & -\beta\overline{S}+\psi\overline{S}+\varphi\\ 0 & \beta G_0-\psi G_0 & \beta\overline{S}-\gamma_2-\psi\overline{S}-\varphi-\delta-\lambda\end{vmatrix}=0$$

$$\Rightarrow\left(-\alpha-\mu-\lambda\right)\left(-\beta G_0-\gamma_1+\psi G_0-\lambda\right)$$
$$\left(\beta\overline{S}-\gamma_2-\psi\overline{S}-\varphi-\delta-\lambda\right)-\left(-\alpha-\mu-\lambda\right)$$
$$\left(\beta G_0-\psi G_0\right)+\left(-\beta\overline{S}+\psi\overline{S}+\varphi\right)-\alpha\gamma_1$$
$$\left(\beta\overline{S}-\gamma_2-\psi\overline{S}-\varphi-\delta-\lambda\right)-\alpha\gamma_2\left(\beta G_0-\psi G_0\right)=0$$

$$\Rightarrow\left(A_1-\lambda\right)\left(A_2-\lambda\right)\left(A_3-\lambda\right)-\left(A_1-\lambda\right)A_4A_5-A_6\left(A_3-\lambda\right)-A_7A_4=0 \tag{18}$$

where, $A_1=-\alpha-\mu,\ A_2=-\beta G_0-\gamma_1+\psi G_0$,

$A_3=\beta\overline{S}-\gamma_2-\psi\overline{S}-\varphi-\delta,\ A_4=\beta G_0-\psi G_0$,

$A_5=-\beta\overline{S}+\psi\overline{S}+\varphi$, and $A_6=\alpha\gamma_1,\ A_7=\alpha\gamma_2$

After simplifying the equation (18), we obtain

$$\lambda^3+\lambda^2\left\{-\left(A_1+A_2+A_3\right)\right\}+\lambda\left\{A_1\left(A_2+A_3\right)A_2+A_3\right\}$$
$$-\lambda A_4A_5-\lambda A_6+A_1A_4A_5-A_1A_2A_3+A_3A_6-A_4A_7=0$$

$$\Rightarrow\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0 \tag{19}$$

where,

$a_1=\left\{-\left(A_1+A_2+A_3\right)\right\}$, $a_2=\left\{A_1\left(A_2+A_3\right)A_2A_3\right\}$ $-A_4A_5-A_6$ and $a_3=A_1A_4A_5-A_1A_2A_3+A_3A_6-A_4A_7$.

Here, $a_1=\left\{-\left(A_1+A_2+A_3\right)\right\}$

$$=\alpha+\mu+\lambda+\beta G_0+\gamma_1-\psi G_0-\beta\overline{S}+\gamma_2+\psi\overline{S}+\varphi+\delta>0$$

similarly, $a_3>0$ and $a_1a_2>a_3$.

From the Routh-Hurwitz criterion [18], we know that all the eigen values of (19) have negative real roots if and only if $a_1>0,\ a_3>0$ and $a_1a_2>a_3$.

Hence, the equilibrium point $\overline{E}_1\left(\overline{A},\overline{S},\overline{G}\right)$ of the model (1)-(3) is asymptotically stable for $a_1>0,\ a_3>0$ and $a_1a_2>a_3$.

3.4.2 Stability at $E_2^*\left(A^*,S^*,G^*\right)$

Theorem 2: The equilibrium point $E_2^*\left(A^*,S^*,G^*\right)$ of the model (1)-(3) is asymptotically stable if $b_1>0,\ b_3>0$ and $b_1b_2>b_3$, otherwise it is unstable [46, 53].

Proof: The Jacobian matrix of the system (1)-(3) is given by

$$J(A,S,G)=\begin{pmatrix}-\alpha-\mu & \gamma_1 & \gamma_2\\ \alpha & -\beta G-\gamma_1+\psi G & -\beta S+\psi S+\varphi\\ 0 & \beta G-\psi G & \beta S-\gamma_2-\psi S-\varphi-\delta\end{pmatrix}$$

At the equilibrium point $E_2^*\left(A^*,S^*,G^*\right)$, the Jacobian matrix takes the following form

$$J\left(A^*,S^*,G^*\right)=\begin{pmatrix}-\alpha-\mu & \gamma_1 & \gamma_2\\ \alpha & -\beta G^*-\gamma_1+\psi G^* & -\beta S^*+\psi S^*+\varphi\\ 0 & \beta G-\psi G & \beta S^*-\gamma_2-\psi S^*-\varphi-\delta\end{pmatrix}$$

The characteristic equation for the eigen value λ is given as

$$\left|J-\lambda I\right|=0$$

$$\begin{vmatrix}-\alpha-\mu-\lambda & \gamma_1 & \gamma_2\\ \alpha & -\beta G^*-\gamma_1+\psi G^*-\lambda & -\beta S^*+\psi S^*+\varphi\\ 0 & \beta G-\psi G & \beta S^*-\gamma_2-\psi S^*-\varphi-\delta-\lambda\end{vmatrix}=0$$

$$\Rightarrow\left(-\alpha-\mu-\lambda\right)\left(-\beta G^*-\gamma_1+\psi G^*-\lambda\right)\begin{pmatrix}\beta S^*-\gamma_2-\psi S^*\\ -\varphi-\delta-\lambda\end{pmatrix}$$
$$-\left(-\alpha-\mu-\lambda\right)\left(\beta G-\psi G\right)\ \left(-\beta S^*+\psi S^*+\varphi\right)-\alpha\gamma_1$$
$$\left(\beta S^*-\gamma_2-\psi S^*-\varphi-\delta-\lambda\right)-\alpha\gamma_2\left(\beta G-\psi G\right)=0$$
$$\Rightarrow\left(B_1-\lambda\right)\left(B_2-\lambda\right)\left(B_3-\lambda\right)-\left(B_1-\lambda\right)B_4B_5$$
$$-B_6\left(B_3-\lambda\right)-B_7B_4=0 \qquad (20)$$

where, $B_1=-\alpha-\mu,\ B_2=-\beta G^*-\gamma_1+\psi G^*$, $B_3=\beta S^*-\gamma_2-\psi S^*-\varphi-\delta,\ B_4=\beta G-\psi G$, $B_5=-\beta S^*+\psi S^*+\varphi,\ \ B_6=\alpha\gamma_1$, and $B_7=\alpha\gamma_2$

After simplifying the equation (20), we get

$$\lambda^3+\lambda^2\left\{-\left(B_1+B_2+B_3\right)\right\}+\lambda\left\{B_1\left(B_2+B_3\right)B_2B_3-B_4B_5-B_6\right\}$$
$$+B_1B_4B_5-B_1B_2B_3+B_3B_6-B_4B_7=0$$
$$\Rightarrow\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0 \qquad (21)$$

where, $b_1=\left\{-\left(B_1+B_2+B_3\right)\right\}$,

$b_2=\left\{B_1\left(B_2+B_3\right)B_2B_3-B_4B_5-B_6\right\}$

and $b_3=B_1B_4B_5-B_1B_2B_3+B_3B_6-B_4B_7$

From the Routh-Hurwitz criterion, we know that all the eigen values of (21) have negative real roots if and only if $b_1>0,\ b_3>0$ and $b_1b_2>b_3$.

By substituting the value of B_1,B_2,B_3,B_4 and B_5, we obtain $b_1>0,\ b_3>0$ and $b_1b_2>b_3$.

Hence, the equilibrium point $E^*\left(A^*,S^*,G^*\right)$ of the model (1)-(3) is asymptotically stable if $b_1>0,\ b_3>0$ and $b_1b_2>b_3$, otherwise it is unstable.

3.4.3 Global Stability at the Equilibrium Point $E_2^*\left(A^*,S^*,G^*\right)$

In this section, we use the Lyapunov direct method [14, 16, 48] to establish sufficient conditions for the global asymptotic stability of the equilibrium point E_2^* in int $\mathbb{R}_+^3$.

Theorem 3: The equilibrium point E_2^* of the of the model (1)-(3) is globally asymptotically stable if $\frac{\alpha\varphi A^*}{\gamma_1 S^*}>\frac{\psi\delta}{\gamma_2 G^*}$ in the interior of the feasible region, otherwise it is unstable.

Proof: Theorem 3 can be proved based on the Lyapunov stability theorem. For that purpose, we consider the following nonlinear Lyapunov function,

$$V=A-A^*-A^*\ln\frac{A}{A^*}+S-S^*-S^*\ln\frac{S}{S^*}+G-G^*$$
$$-G^*\ln\frac{G}{G^*}$$
(22)

Then V is C^1 on the interior of, $\mathbb{R}_+^3$ $E_2^*\left(A^*,S^*,G^*\right)$ is the equilibrium point.

The derivative of (22) along the solution curves of the model (1)-(3) is given by the expression:

$$\dot V=\dot A-\frac{A^*\dot A}{A}+\dot S-\frac{S^*\dot S}{S}+\dot G-\frac{G^*\dot G}{G}$$
$$=\gamma_1S+\gamma_2G-\alpha A-\mu A-\frac{A^*}{A}\left(\gamma_1S+\gamma_2G-\alpha A-\mu A\right)+$$
$$\alpha A-\beta GS-\gamma_1S+\psi GS+\varphi G-\frac{S^*}{S}\begin{pmatrix}\alpha A-\beta GS-\gamma_1S\\ +\psi GS+\varphi G\end{pmatrix}$$
$$+\beta GS-\gamma_2G-\psi GS-\varphi G-\delta G-\frac{G^*}{G}\begin{pmatrix}\beta GS-\gamma_2G\\ -\psi GS-\varphi G-\delta G\end{pmatrix}$$
$$=\gamma_1S\left(1-\frac{A^*}{A}\right)+\gamma_2G\left(1-\frac{A^*}{A}\right)-(\alpha+\mu)A\left(1-\frac{A^*}{A}\right)+$$
$$\alpha A\left(1-\frac{S^*}{S}\right)-(\beta-\psi)GS\left(1-\frac{S^*}{S}\right)$$
$$-\gamma_1S\left(1-\frac{S^*}{S}\right)+\varphi G\left(1-\frac{S^*}{S}\right)-(\gamma_2+\varphi+\delta)$$
$$G\left(1-\frac{G^*}{G}\right)+(\beta-\psi)GS\left(1-\frac{G^*}{G}\right)$$
(23)

At the equilibrium point $E_2^*\left(A^*,S^*,G^*\right)$, we have

$$\alpha+\mu=\frac{\gamma_1 S^*+\gamma_2 G^*}{A^*} \tag{24}$$

$$\beta-\psi=\frac{\alpha A^*-\gamma_1 S^*+\varphi G^*}{G^* S^*} \tag{25}$$

$$\gamma_2+\varphi+\delta=\frac{\beta G^* S^*-\psi G^* S^*}{G^*} \tag{26}$$

using the equations (24)-(26) in (23), we obtain

$$\dot{V}=\gamma_1 S\left(1-\frac{A^*}{A}\right)+\gamma_2 G\left(1-\frac{A^*}{A}\right)-\frac{\gamma_1 S^*+\gamma_2 G^*}{A^*}$$
$$A\left(1-\frac{A^*}{A}\right)+\alpha A\left(1-\frac{S^*}{S}\right)-\frac{\alpha A^*-\gamma_1 S^*+\varphi G^*}{G^* S^*}$$
$$GS\left(1-\frac{S^*}{S}\right)-\gamma_1 S\left(1-\frac{S^*}{S}\right)+\varphi G\left(1-\frac{S^*}{S}\right)-$$
$$\frac{\beta G^* S^*-\psi G^* S^*}{G^*}G\left(1-\frac{G^*}{G}\right)+\frac{\alpha A^*-\gamma_1 S^*+\varphi G^*}{G^* S^*}$$
$$GS\left(1-\frac{G^*}{G}\right)$$
$$=\gamma_1 S-\frac{\gamma_1 S A^*}{A}+\gamma_2 G-\frac{\gamma_2 G A^*}{A}-\frac{\left(\gamma_1 S^*+\gamma_2 G^*\right)A}{A^*}$$
$$+\left(\gamma_1 S^*+\gamma_2 G^*\right)+\alpha A-\frac{\alpha A S^*}{S}-\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)GS}{G^* S^*}$$
$$+\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)G}{G^*}-\gamma_1 S^*+\varphi G\frac{S^*}{S}-$$
$$\frac{\beta G^* S^*-\psi G^* S^*}{G^*}G+\beta G^* S^*-\psi G^* S^*+$$
$$\frac{\alpha A^*-\gamma_1 S^*+\varphi G^*}{G^* S^*}GS-\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)S}{S^*}$$
$$=\gamma_1\left(S-S^*\right)-\frac{\gamma_1 S A^{*2}-\left(\gamma_1 S^*+\gamma_2 G^*\right)A^2}{AA^*}-\frac{\gamma_2 G A^*}{A}+$$
$$\gamma_1 S^*+\alpha A-\frac{\alpha A S^*}{S}-\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)GS}{G^* S^*}$$
$$+\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)G-\beta G G^* S^*-\psi G G^* S^*}{G^*}$$
$$+\gamma_2\left(G-G^*\right)+\beta G^* S^*-\psi G^* S^*+\frac{\alpha A^*-\gamma_1 S^*+\varphi G^*}{G^* S^*}GS$$
$$-\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)S^2+\varphi G S^{*2}}{SS^*}$$

$$=\gamma_1\left(S-S^*\right)-\frac{\gamma_2 G A^*}{A}+\gamma_1 S^*-\frac{\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)GS+}{G^* S^*}$$
$$\left(\left(\alpha A^*-\gamma_1 S^*+\varphi G^*\right)S^* G-\beta G G^* S^{*2}-\psi G G^* S^{*2}\right)\frac{1}{G^* S^{*2}}+$$
$$\gamma_2\left(G-G^*\right)+\beta G^* S^*-\psi G^* S^*-\frac{\alpha\varphi\gamma_1 A^* S^* S^2 G S^{*2}}{SS^*}\frac{AG^* S}{S^* G A^*}$$
$$-\frac{\gamma_1 S A^{*2}-\left(\gamma_1 S^*+\gamma_2 G^*\right)A^2}{AA^*}+\alpha A$$
$$-\frac{\alpha A S^*}{S}+\frac{\alpha\varphi\gamma_1 A^* S^* S^2 G S^{*2}}{SS^*}\left(4-\frac{A^*}{A}-\frac{S^* G}{G^* S}\right)$$

$$=\gamma_1\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)-\frac{\gamma_2 G A^*}{A}-\gamma_1 S^*-\alpha A\frac{\left(A^*+\gamma_1 S^*+\varphi G^*\right)}{G^* S^*}$$
$$\left(\frac{\alpha\varphi A^*}{\gamma_1 S^*}-\frac{\psi\delta}{\gamma_2 G^*}\right)-\frac{\gamma_1 S A^{*2}-\left(\gamma_1 S^*+\gamma_2 G^*\right)A^2}{AA^*}$$
$$+\gamma_2\left(G-G^*\right)-\beta G^* S^{**}+\frac{\alpha\varphi\gamma_1 A^* S^* S^2 G S^{*2}}{SS^*}$$
$$\left(4-\frac{A^*}{A}-\frac{S^* G}{G^* S}-\frac{AG^* S}{S^* G A^*}\right)$$

Since the arithmetic mean is greater than or equal to the geometric mean and geometric mean is greater than or equal to the harmonic mean, it follows that

$$2-\frac{S^*}{S}-\frac{S}{S^*}\le 0$$
$$4-\frac{A^*}{A}-\frac{S^* G}{G^* S}-\frac{AG^* S}{S^* G A^*}\le 0$$

Further, since all the model parameters are nonnegative, it follows that $\dot{V}\le 0$ for $\frac{\alpha\varphi A^*}{\gamma_1 S^*}>\frac{\psi\delta}{\gamma_2 G^*}$ with $\dot{V}=0$ if and only if $A=A^*, S=S^*$ and $\mathrm{G}=G^*$ holds.

The largest compact invariant set in $\left\{(A,S,G)\in\mathbb{R}_+^3 : dV/dt=0\right\}$ is the singleton $\left\{E_2^*\right\}$, where E_2^* is the equilibrium point of the model. By using LaSalle's invariance principle [37, 54] then implies that $E_2^*\left(A^*,S^*,G^*\right)$ is globally asymptotically stable in the interior of $\mathbb{R}_+^3$ if $\frac{\alpha\varphi A^*}{\gamma_1 S^*}>\frac{\psi\delta}{\gamma_2 G^*}$, otherwise it is unstable.

4 Numerical analysis

In this section, we have analyzed the model with graphical analysis and sensitivity analysis.

4.1 Numerical simulations

We have solved the model numerically based on the respective parameters present in the system of equations

(1)-(3) to investigate the dynamical behavior of the model. The simulations are carried out by ode45 solver using MATLAB programming language. We use a set of suitable parameter values. The description of all the parameters with the estimated values used in the simulation is presented in Table 1. We have considered the initial condition $A_0 = 50 \times 10^9$, $S_0 = 110 \times 10^9$ and $G_0 = 60 \times 10^9$.

Firstly, we solve the model (1)-(3) considering the initial values and all other parameters that are estimated from [6, 56]. Also, we have performed the numerical simulations for time interval $t \in [0, 20]$ for 20 years. The value of the model parameters is given in Table 1.

Table 1: Values and explanation of parameters

Descriptions	Symbols	Values
Precipitation rate	α	0.5
Dissipation rate	μ	0.01
Infiltration rate	β	0.3005
Deforestation rate	δ	0.06
Pumping rate of water	φ	0.02
Pollution rate	ψ	0.300
Evaporation rate from surface water	γ_1	0.09
Evaporation rate from ground water level	γ_2	0.028

Our object is to study the effects of deforestation rate (δ), pumping rate of water (φ), pollution rate (ψ) and evaporation rate (γ_2) due to global warming on groundwater level as well as in surface water and atmospheric water. We have selected these parameters because they have a large impact on groundwater level declination. So, if it is possible to minimize the deforestation rate, pumping rate of water, pollution rate then the declination of groundwater level will be controlled. Considering these parameters into account, we have run the program for the state variables, atmospheric water $A(t)$, surface water $S(t)$ and groundwater level $G(t)$). The result of simulation of the combined class is presented in Figure 3.

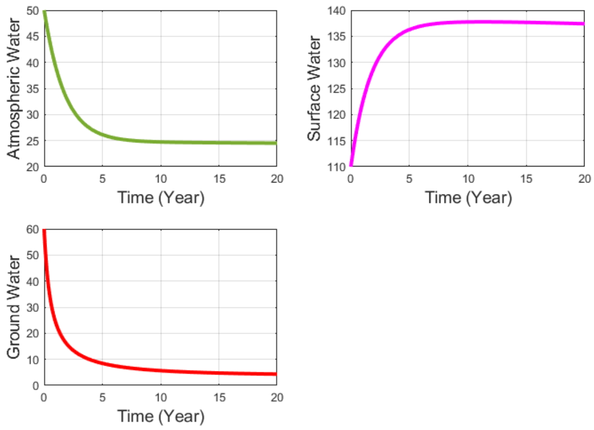

Figure 3: Numerical simulation for Groundwater model, with time (20 years).

From **Figure 3**, we observe that groundwater level is decreasing day by day due to natural phenomena, over demand, natural disaster and so on whereas surface water primarily increases but after sometimes it shows the stable situation. At the same time, atmospheric water is also decreasing day by day. In everyday life we would face the problems with freshwater shortage is sure to cause problems in many aspects of our lives. The activities that lead to groundwater declination come mostly from human being, but a portion of it also comes from changes in our climate (for examples, deforestation, drought, global warming etc.) and can speed up the process. So, the human population is the main responsible for water contamination.
Again, we run the program for two state variables, surface water and groundwater level keeping all the values of the parameters same as before. The result obtained in this case is represented in Figure 4.

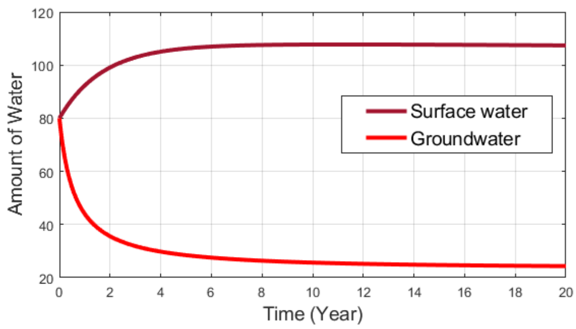

Figure 4: Dynamics of surface water and groundwater where groundwater decrease significantly as a result surface water is increasing day by day.

Figure 4 shows the state trajectories of the two compartments such as surface water and groundwater in the absence

of any control measures. We have observed that when no control measure is employed, groundwater level decreases from initial state. At the same time, surface water gradually increases from the initial position and after some years it reaches to the peak level. It is occurred due to evaporation of water from the ocean and land surfaces, because this water is temporarily considered as vapor in the atmosphere, and falls back to earth's surface as precipitation, then surface water is formed by the residue of precipitation and melted snow and then it will be stable if no others factor (for examples, drought and low streamflow or over streamflow) as employed in the atmosphere.

Now, we run the program keeping all other values of the parameters same as before. We have some small changed to pollution rate to show the effect of pollution rate on groundwater level declination, we consider the same initial values of the three state variables. The result obtained in this case is given in Figure 5.

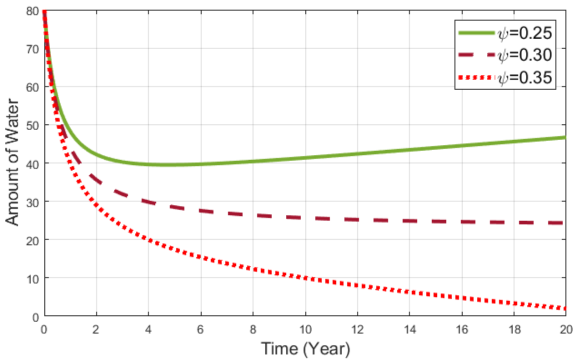

Figure 5: Variation of groundwater for different values of pollution rate where groundwater level is significantly decreased due to increase of pollution rate.

In Figure 5, we see the variation of groundwater level for different values of pollution rate (ψ) with time. It is easy to see that falling of groundwater level is increased surprisingly due to increase of pollution rate ψ. In other words, falling of groundwater level decreases for the lower value of pollution rate ψ. Groundwater protection specialists inspect and analyze that human are the main cause of water contamination, which is provoked in many ways: by the dumping of industrial waste; due to temperature rise, that cause the variation of water by reducing the oxygen in its composition; which causes of silt and bacteria to appear under the soil and therefore pollute groundwater and is declined significantly in day by day. Thus, groundwater level decreases as the increase of pollution rate and groundwater level increases as the decrease of pollution rate.

Again, we solve the model numerically for the case of groundwater to show the change in the groundwater level due to frequent pumping of water. Taking the values of all parameters into account as in Table 1 and considering the same initial values of the state variables as we did before. The result in this case is presented in Figure 6.

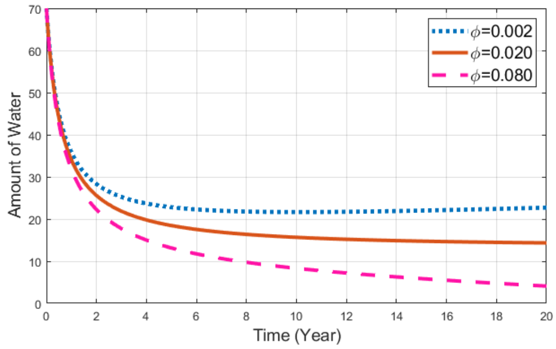

Figure 6: Variation of groundwater for different values of pumping rate were falling of groundwater level is significantly increased due to increase of pumping of water.

Figure 6 represents the variation of groundwater level for different values of pumping rate φ with time while pollution rate and other effects are fixed. We observe that falling of groundwater level is increased tremendously due to increase of pumping rate φ. The more we extract groundwater from below the earth's surface, then we have to go in the downward of porous soil and rock in order to get more fresh water. But it is the matter of very anxious that when we have emitted water from deeper within the Earth, we find that there is less water available. Consequently, we will have to use more resources to develop alternative methods to reach further into the groundwater level.

Next, we have solved the model numerically to show the change in the groundwater level due to evaporation rate for the parameters in Table. The result in this case is shown in Figure 7.

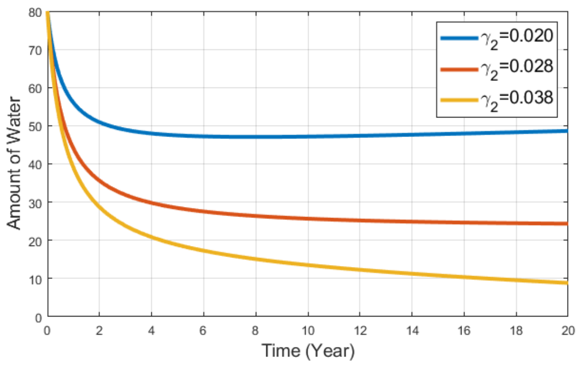

Figure 7: Variation of groundwater for different values of evaporation rate where falling of groundwater level is extensively increased due to global warming.

In **Figure 7**, one can see the variation of groundwater level for different values of evaporation rate (γ_2) over the time while deforestation rate, pollution rate and other effects are constant. It is easy to understand that falling of groundwater level is increased significantly due to increase of evaporation rate γ_2. That is, falling of groundwater level decreases for the lower value of evaporation rate γ_2 and falling of groundwater level increases for the higher value of evaporation rate γ_2. Global warming can lead to longer periods of droughts, which directly affects availability and dependency on groundwater. Moreover, due to long periods of droughts there is a higher risk of declination on aquifers, especially in case of small and shallow aquifers as a result groundwater level is declined.

Lastly, we run the program keeping all other values of the parameters same as before and change in the deforestation rate to show the effect of deforestation on groundwater level. The result in this case is presented in Figure 8.

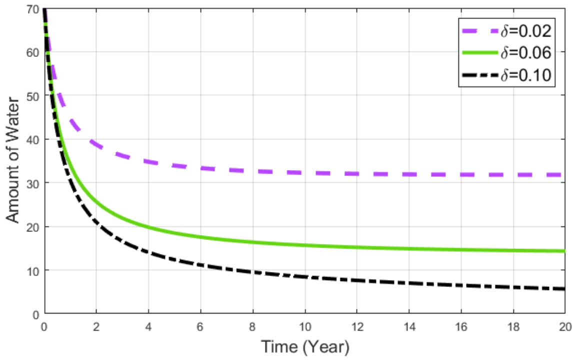

Figure 8: Groundwater level is tremendously decreased due to increase of deforestation rate from $\delta = 0.02$ to $\delta = 0.10$

Figure 8 represents the variation of groundwater level for different values of deforestation rate δ with time while evaporation rate, pumping rate and other effects are constant. We observe that groundwater level is decreased extensively due to increase of deforestation rate δ. In other words, falling of groundwater level decreases for the lower value of deforestation rate δ and falling of groundwater level increases for the higher value of deforestation rate δ. The diminution of trees and other plants can cause global warming, soil erosion, desertification, fewer crops, flooding, increased greenhouse gases in the environment. Thus, it has a significant impact on groundwater. The roots from vegetation and plantation are used to filter the various contamination from groundwater. Thus, by increasing deforestation i.e. by reducing plantation and vegetation, groundwater is faced a great threat in our country as well as all over the world.

4.2 Sensitivity analysis

To evaluate the proper management of groundwater and realizing the impact of climate change or global environmental crisis on groundwater level declination in Bangladesh as well as all over the world, it is mandatory to know the relative importance of each input parameter.

In Sensitivity analysis, we evaluate the uncertainty in the output of a mathematical simulation and which parameters and interactions have the most significant impact on the dynamical behavior of the system. It plays a significant role to the experimental designs and data assimilation of nonlinear compartmental model [8].

There are many ways to perform the sensitivity analysis. One of the ways is normalized forward sensitivity index of a variable with respect to a parameter. It is defined as the ratio of the relative change in the variable to the comparative change in the parameter [17]. When the variable is a differentiable function of the parameter, then the sensitivity index may be represented with partial derivatives.

If the normalized forward sensitivity index of G^* is differentiable with respect to a given parameter P, then it is denoted as $\gamma_P^{G^*}$ and defined as

$$\gamma_P^{G^*} = \frac{\partial G^*}{\partial P}\frac{P}{G^*}$$

For example, the sensitivity index of G^* with respect to β is

$$\gamma_\beta^{G^*} = \frac{\partial G^*}{\partial \beta}\frac{\beta}{G^*} = 0.2763$$

The values of the sensitivity indices for the corresponding parameters that we have used in the model are presented in Table 2.

Parameters	Values	Sensitivity index
β	0.3005	0.2763
δ	0.06	-0.7203
φ	0.02	-0.8462
ψ	0.300	-0.4860
γ_2	0.028	-0.0312

Table 2: Sensitivity indices of different parameter values given in Table 1

From Table 2, we notice that the most sensitive parameters to the groundwater level declination of the model (1)-(3) are infiltration rate (β), deforestation rate (δ), pumping rate of water (φ), and pollution rate (ψ). In practically, the sensitivity analysis predicts that an increase of the infiltration rate β will increase the groundwater level by 27.63%. Conversely, an increase of the value of frequently pumping rate of water φ will decrease groundwater level by 84.62%. These phenomena also can be observed from the **Figures 9-12**.

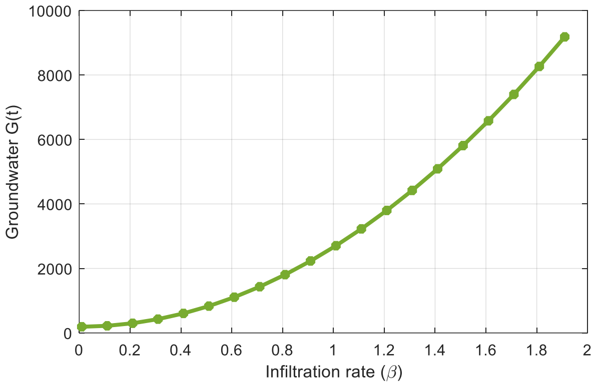

Figure 9: Sensitivity graph of groundwater level on the basis of infiltration rate (β).

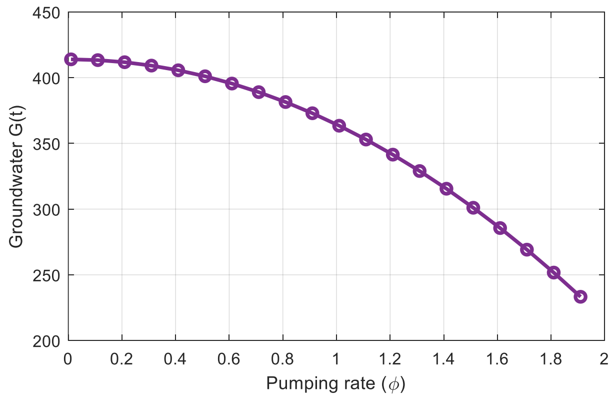

Figure 10. Sensitivity graph of groundwater level on the basis of pumping rate (φ)

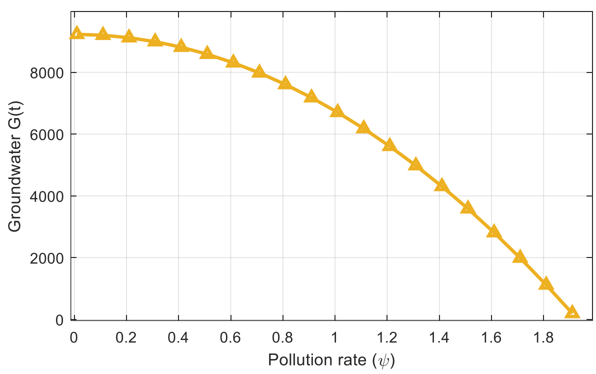

Figure 11: Sensitivity graph of groundwater level on the basis of pollution rate (ψ).

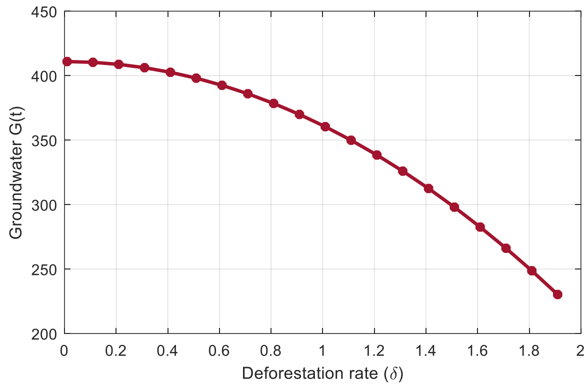

Figure 12: Sensitivity graph of groundwater level on the basis of deforestation rate (δ).

5 Phase portrait analysis

We now investigate the dynamical behavior of the model (1)-(3) by plotting phase-plane diagram based on the respective parameters present in the model. In these phase-planes, small arrows in Figures 9-11 show the direction field; the red dot represents the equilibrium point and dashed lines are the nullclines.

In this model, we have computed three graphs (see **Figures 13-15**) to show the stability of the model near equilibrium points. It is clear that the pollution free equilibrium point is asymptotically stable and the nature of equilibrium is node for the parameter values $\alpha = .05,\ \beta = 0.3005,\ \varphi = 0.02, \psi = 0.300$ and $\alpha = .05,\ \beta = 0.3005,\ \varphi = 0.06,\ \psi = 0.400$ but when the value of pollution rate and frequent pumping rate is continuously increased, then the solution curve of the groundwater level is extensively decreased and the nature of equilibrium is unstable saddle point.

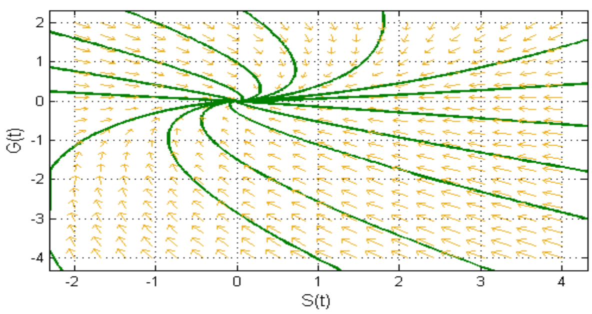

Figure 13: Phase plane for the model (1)-(3) with the parameter values
$\alpha = .05,\ \beta = 0.3005,\ \varphi = 0.02$ and $\psi = 0.300$

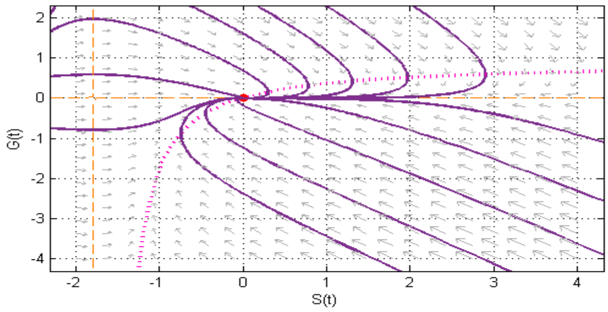

Figure 14: Phase plane and nullclines for the model (1)-(3) with the parameter values $\alpha=.05,\ \beta=0.3005,\ \varphi=0.06$ and $\psi=0.400$

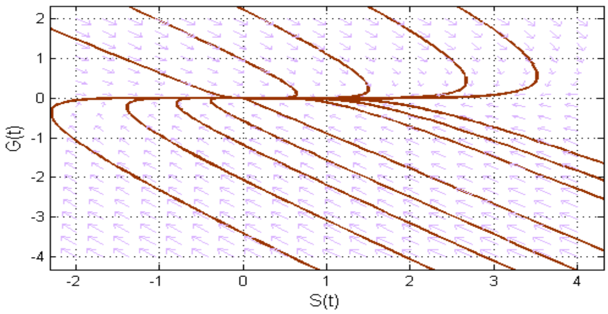

Figure 15: Phase plane for the model (1)-(3) with the parameter values $\alpha=.05,\ \beta=0.3005,\ \varphi=0.500$ and $\psi=0.300$

6 Conclusions

In this study, a mathematical model on groundwater level declination is presented with qualitative and quantitative analysis. We investigate the dynamical behavior of the model (1)-(3) by stability analysis and plotting phase-plane diagram based on the respective parameters present in the model. From the numerical simulations, it is clear from Figures 5 and 6 that, the groundwater level is significantly decreased due to increase of pollution rate while the falling of groundwater level is tremendously increased due to increase of over pumping through shallow tubewells, deep tubewells and low lift pumps. Because excessive pumping is the cause of lower groundwater table. Again, Groundwater and surface water are connected to each other. When groundwater is used superfluously, then lakes, streams and rivers that are connected to subsurface water can also have their supply diminished. We also observe from Figures 7 and 8 that, Groundwater level is tremendously decreased due to increase of deforestation rate, where falling of groundwater level is extensively increased for higher rate of evaporation that is caused of global warming.

Lastly, we demonstrate that, this model gives a latest picture of groundwater level management in Bangladesh as well as all over the world that we have to very careful and conscious to use water specifically groundwater and making sure that we have to reduce the misusage of it in our daily life. The proposed model can be helped for the researchers and planners who are associated with the research of groundwater level. It also may be helpful for the government to make and take decision regarding the prevention of groundwater level declination as well as may be increase the public awareness in case of using groundwater.

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Conflicts of Interest:

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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