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Some Picture Fuzzy Aggregation Operators Based on Frank t-norm and t-conorm: Application to MADM Process

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In this paper, we develop some new operational laws and their corresponding aggregation operators for picture fuzzy sets (PFSs). The PFS is a powerful tool to deal with vagueness, which is a generalization of a fuzzy set and an intuitionistic fuzzy set (IFS). PFSs can model uncertainty in situations that consist of more than two answers like yes, refusal, neutral, and no. The operations of t-norm and t-conorm, developed by Frank, are usually a better application with its flexibility. From that point of view, the concepts of Frank t-norm and t-conorm are introduced to aggregate picture fuzzy information. We propose some new operational laws of picture fuzzy numbers (PFNs) based on Frank t-norm and t-conorm. Further, with the assistance of these operational laws, we have introduced picture fuzzy Frank weighted averaging (PFFWA) operator, picture fuzzy Frank order weighted averaging (PFFOWA) operator, picture fuzzy Frank hybrid averaging (PFFHA) operator, picture fuzzy Frank weighted geometric (PFFWG) operator, picture fuzzy Frank order weighted geometric (PFFOWG) operator, picture fuzzy Frank hybrid geometric (PFFHG) operator and discussed with their suitable properties. Then, with the help of PFFWA and PFFWG operators, we have presented an algorithm to solve multiple-attribute decision making (MADM) problems under the picture fuzzy environment. Finally, we have used a numerical example to illustrate the flexibility and validity of the proposed method and compared the results with other existing methods.

Povzetek: Prispevek se ukvarja z operatorji mehkih množic na osnovi Frankove t-norme in t-konorme.

1 Introduction

In real-life situations, the fuzzy set theory [48] plays a vital role in handling the vagueness of human choices. Then continuous efforts are paid for further generalization of fuzzy set theory. The IFS theory is one of such generalizations, introduced by Atanssov [2]. IFS is characterized by a degree of membership and degree of non-membership such that their sum does not exceed one. However, IFSs are insufficient to handle the possibility with more than two answers as just yes-no type. Consider the case of usual voting where one has the choices like a vote for, vote against, abstain from voting, and refusal. To deal with such situations with high accuracy, Cuong and Krienovich [8] conveyed a novel concept of picture fuzzy set (PFS). PFS is characterized by a membership degree, a non-membership degree, and a neutrality degree such that their sum is less than or equal to one. Cuong [9] examined few properties of PFSs and introduced distance measures between PFSs.

Recently, some research models have been developed on the picture fuzzy (PF) environment. Dinh and Thao [10] introduced some distance measures and dissimilarity measures between PFSs and applied them to MADM problems. Wang and Li [33] proposed a hesitant fuzzy set in the PF environment and developed picture hesitant fuzzy aggregation operators. Wei [39] extended the TODIM method to MADM problems under the PF environment. Wei [40] developed some similarity measures between PFSs such as cosine measure, set-theoretic cosine similarity measure, grey similarity measure and applied these to building material recognition and mineral field recognition. Dutta and Ganju [11] introduced decomposition theorems of PFSs and defined the extension principle for PFSs. Wei [35] introduced PF cross-entropy as an extension of the cross-entropy of fuzzy sets. Xu et al. [45] developed some aggregation operators for fusing PF information. Dutta [12] applied distance measure between PFSs in medical diagnosis. Singh [27] proposed correlation coefficients for PFSs and gave the geometrical interpretation for PFSs. Son [28] and Thong [29] introduced several clustering algorithms with PFSs. Le et al. [21] proposed some dissimilarity measures under PF information and applied them to decision-making problems. Wei et al. [36] introduced the projection models for the MADM problem with PF information. Wei and Gao [42] developed the generalized dice similarity measure under PF environment and applied them to building material recognition. Zeng et al. [50] proposed the exponential Jensen PF divergence measure and applied it in multi-criteria group decision making. Several researchers proposed information aggregation operators under the PF environment [3, 16, 17, 22, 30, 32, 43, 51]. Garg [14] presented some PF aggregation operators and applied them to multi-criteria decision making. Wei [38] presented cosine similarity measures for PFS and applied them to strategic decision making. Wei [41] proposed PF Hamacher aggregation operators and applied them to the MADM process. Khan et al. [18] investigated the information aggregation operators method under the PF environment with the help of Einstein norm operations. Khan et al. [19] introduced a series of logarithmic aggregation operators under the PF environment. Wang et al. [34] proposed Muirhead mean operators under PF environment and applied them to evaluate the financial investment risk.

A fascinating generalization of probabilistic and Lukasiewicz t-norm and t-conorm [23] are Frank t-norm and t-conorm [13], which form an ordinary and adequately flexible family of the continuous triangular norm. The employment of a specific parameter makes the Frank t-norm and t-conorm more resilient along with the procedure of fusion of information. Several works [1, 20] can be found in the literature related to Frank t-norm and t-conorm. The functional equations of Frank and Alsina are thoroughly studied by Calvo et al. [4] for two classes containing commutative, associative, and increasing binary operators. Exploring the additive generating function (AGF) of Frank t-norms, Yager [46] launched a framework in approximate reasoning with Frank t-norms. Casasnovas and Torrens [5] introduced a novel axiomatic approach to the scalar cardinality of Frank t-norms, and they further established the properties of other standard t-norms. Comparing between the Frank t-norms and the Hamacher t-norms up to an extent, Sarkoci [26] concluded that two different t-norms belong to the same family. Xing et al. [44] introduced aggregation operators for Pythagorean fuzzy set based on Frank t-norm and t-conorm and then applied them to solve MADM problems. Zhou et al. [52] investigated some Frank aggregation operators of interval-valued neutrosophic numbers and analyzed a case study of selecting agriculture socialization. Qin and Liu [24] introduced Frank aggregation operators for a triangular interval type-2 fuzzy set and applied it to solve multiple attribute group decision making (MAGDM) problems. Qin et al. [25] developed some hesitant fuzzy aggregation operators based on Frank t-norm operations.

Evidently, a general t-norm and t-conorm can be used for shaping both the intersection and union of PFS. The PFS is compatible to reveal uncertain information. Since the Frank aggregation operators involve a parameter so the operators make the information process more flexible and strong. The investigation on the applications of Frank operators is rare, specifically in the area of information aggregation and decision making. Keeping this in mind, it is worthy to prolong Frank t-norm and t-conorm to handle the PF environment. With such motivation of aforementioned analysis, we have introduced new operational rules of PFNs based on Frank operators and exhibited their characteristics.

In this paper, we have introduced some new operational laws for PFNs based on Frank t-norm and Frank t-conorm. Then using these operational laws, we have developed Frank t-norm and t-conorm based PFFWA, PFFOWA, PFHWA, PFFWG, PFFOWG, and PFFHG aggregation operators. We have also investigated some of their desirable properties. Utilizing PFFWA and PFFWG operators, we have developed an algorithm to solve an MADM problem under the PF environment. To illustrate the validity and superiority of the proposed method, a numerical example is considered, solved, and the obtained results are compared with other existing well-known methods.

The rest of the paper is organized as follows.

In Section 2, some basic definitions and preliminaries are recalled, which help us to make the concept about the present article. In Section 3, some new operational laws for PFNs based on Frank t-norm and t-conorm have been proposed, and using those operational laws, some new aggregation operators are defined in the PF environment. An algorithm to solve the decision-making problems based on Frank aggregation operators is presented in Section 4. In Section 5, we have checked the validity of the proposed method through a real-life example. Section 6 analyze the effect of the parameters on the decision-making result. Section 7 presents a useful comparison between the results of our proposed method and other significant models. Finally, the conclusion is made in Section 8.

2 Preliminaries

In this section, we recall some basic definitions and preliminaries.

Definition 2.1. [6, 7] Let us consider X as a universal set. The PFS $\tilde{P}$ over the universal set X is interpreted as

$$\tilde{P} = \{\langle x, \mu_{\tilde{P}}(x), \eta_{\tilde{P}}(x), \nu_{\tilde{P}}(x)\rangle | x \in X\}$$

where $\mu_{\tilde{P}} : X \to [0,1]$, $\eta_{\tilde{P}} : X \to [0,1]$ and $\nu_{\tilde{P}} : X \to [0,1]$ are called the positive degree of membership, neutral degree of membership and the negative degree of membership to the set $\tilde{P}$ respectively, with the condition $0 \leq \mu_{\tilde{P}}(x) + \eta_{\tilde{P}}(x) + \nu_{\tilde{P}}(x) \leq 1$ for every $x \in X$. Also the degree of hesitancy for $x \in X$ is defined as $\pi_{\tilde{P}}(x) = 1-\mu_{\tilde{P}}(x)-\eta_{\tilde{P}}(x)-\nu_{\tilde{P}}(x)$. For our convenience, we denote $p = (\mu_p, \eta_p, \nu_p)$ as a picture fuzzy number (PFN).

Definition 2.2. [6, 37] Let $p = (\mu_p, \eta_p, \nu_p)$ and $q = (\mu_q, \eta_q, \nu_q)$ be two PFNs over the universal set X and $\xi > 0$ be any real number, then the corresponding operations are defined as follows:

1. $p \leq q$, if $\mu_p \leq \mu_q$, $\eta_p \leq \eta_q$ and $\nu_p \geq \nu_q$
2. $p \bigvee q$=$(max\{\mu_p, \mu_q\}, min\{\eta_p, \eta_q\}, min\{\nu_p, \nu_q\})$.
3. $p \bigwedge q$=$(min\{\mu_p, \mu_q\}, max\{\eta_p, \eta_q\}, max\{\nu_p, \nu_q\})$.
4. $p^c = (\nu_p, \eta_p, \mu_p)$.
5. $p \wedge q$=$(min\{\mu_p, \mu_q\}, max\{\eta_p, \eta_q\}, max\{\nu_p, \nu_q\})$.
6. $p \vee q$=$(max\{\mu_p, \mu_q\}, min\{\eta_p, \eta_q\}, min\{\nu_p, \nu_q\})$.
7. $p \oplus q = (\mu_p + \mu_q - \mu_p\mu_q, \eta_p\eta_q, \nu_p\nu_q)$.
8. $p \otimes q = (\mu_p\mu_q, \eta_p + \eta_q - \eta_p\eta_q, \nu_p + \nu_q - \nu_p\nu_q)$.
9. $\xi p = (1 - (1 - \mu_p)^\xi, \eta_p^\xi, \nu_p^\xi)$.
10. $p^\xi = (\mu_p^\xi, 1 - (1 - \eta_p)^\xi, 1 - (1 - \nu_p)^\xi)$.

Definition 2.3. [16] The score function of the PFN $p = (\mu_p, \eta_p, \nu_p)$ is defined by

$$\Delta(p) = \frac{1 + \mu_p - \nu_p}{2}$$

where $\Delta(p) \in [0, 1]$.

Definition 2.4. [16] The accuracy function of the PFN $p = (\mu_p, \eta_p, \nu_p)$ is defined by

$$\nabla(p) = \mu_p + \nu_p$$

where $\Psi(p) \in [-1, 1]$.

According to Definitions 2.3 and 2.4, if $p = (\mu_p, \eta_p, \nu_p)$ and $q = (\mu_q, \eta_q, \nu_q)$ be any two PFNs then

1. If $\Delta(p) > \Delta(q)$ then $p > q$,
2. If $\Delta(p) < \Delta(q)$ then $p < q$,
3. If $\Delta(p) = \Delta(q)$, then
   - If $\nabla(p) > \nabla(q)$, then $p > q$,
   - If $\nabla(p) = \nabla(q)$, then $p = q$.

Wei [37] introduced the PF aggregation operators depicted in the upcoming definitions.

Definition 2.5. Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})$ $(i = 1, 2, \dots, n)$ be a number of PFNs. Then the aggregated value of them using PF weighted averaging (PFWA) operator is also a PFN and $PFWA(p_1, p_2, \dots, p_n) = \bigoplus_{i=1}^{n}(w_i p_i) = \left(1 - \prod_{i=1}^{n}(1 - \mu_{p_i})^{w_i}, \prod_{i=1}^{n}{\eta_{p_i}}^{w_i}, \prod_{i=1}^{n}{\nu_{p_i}}^{w_i}\right)$, where $w = (w_1, w_2, \dots, w_n)^t$ be the weight vector of $p_i (i = 1, 2, \dots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 2.6. Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})$ $(i = 1, 2, \dots, n)$ be a number of PFNs. The PF order weighted averaging (PFOWA) operator of dimension n is a function $p^n \to p$ such that, $PFOWA(p_1, p_2, \dots, p_n) = \bigoplus_{i=1}^{n}(w_i p_{\rho(i)}) = \left(1 - \prod_{i=1}^{n}(1 - \mu_{p_{\rho(i)}})^{w_i}, \prod_{i=1}^{n}{\eta_{p_{\rho(i)}}}^{w_i}, \prod_{i=1}^{n}{\nu_{p_{\rho(i)}}}^{w_i}\right)$, where $w = (w_1, w_2, \dots, w_n)^t$ be the weight vector of $p_i (i = 1, 2, \dots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, $(\rho(1), \rho(2), \dots, \rho(n))$ is the permutation of $(i = 1, 2, \dots, n)$, for which $p_{\rho(i-1)} \geq p_{\rho(i)}$ for all $i = 1, 2, \dots, n$.

In the following, we recall the definition of Frank t-norm and t-conorm.

Definition 2.7. [13] Let us assume that a and b be two real numbers. Then, Frank t-norm and Frank t-conorm are defined by,

$Fra(a,b)$=$\log_r\left(1 + \frac{(r^a - 1)(r^b - 1)}{r - 1}\right)$

$Fra'(a,b)$=$1 - \log_r\left(1 + \frac{(r^{1-a} - 1)(r^{1-b} - 1)}{r - 1}\right)$

respectively, where $(a, b) \in [0, 1] \times [0, 1]$ and $r \neq 1$.

Based on limit theory, we observe some interesting results [31]:

1. If $r \to 1$, then $Fra'(a,b) \to a + b - ab$ and $Fra(a,b) \to ab$. Therefore, if r tends to 1 the the Frank sum and Frank product reduced to the probabilistic sum and probabilistic product.

2. If $r \to \infty$, then $Fra'(a,b) \to min\{a + b, 1\}$ and $Fra(a,b) \to max\{0, a + b - 1\}$. So, for r tends to infinity the Frank sum and the Frank product reduced to the Lukasiewicz sum and Lukasiewicz product.

EXAMPLE **1.** Let $a = 0.29$, $b = 0.56$ and $r = 4$, then,
$Fra(0.29, 0.56) = \log_4\left(1 + \frac{(4^{0.29}-1)(4^{0.56}-1)}{4-1}\right)$
=0.1276.
$Fra'(0.29, 0.56)$=
$1 - \log_4\left(1 + \frac{(4^{1-0.29}-1)(4^{1-0.56}-1)}{4-1}\right)$=0.8723.

3 Picture fuzzy Frank aggregation operators

In this section, we develop some operational rules under the PF environment with the assistance of Frank t-norm and t-conorm. Further, we propose the PFFWA, PFFOWA, PFFHWA, PFFWG, PFFOWG and PFFHWG aggregation operators using our developed operational rules.

Definition 3.1. Let $p = (\mu_p, \eta_p, \nu_p)$, $p_1 = (\mu_{p_1}, \eta_{p_1}, \nu_{p_1})$ and $p_2 = (\mu_{p_2}, \eta_{p_2}, \nu_{p_2})$ be any three PFNs, $r > 1$ and $\xi > 0$ be any real number. Then Frank t-norm and t-conorm operations of PFNs are defined as:

1. $p_1 \oplus p_2 = \left(1 - \log_r\left(1 + \frac{(r^{1-\mu_{p_1}} - 1)(r^{1-\mu_{p_2}} - 1)}{r - 1}\right), \log_r\left(1 + \frac{(r^{\eta_{p_1}} - 1)(r^{\eta_{p_2}} - 1)}{r - 1}\right), \log_r\left(1 + \frac{(r^{\nu_{p_1}} - 1)(r^{\nu_{p_2}} - 1)}{r - 1}\right)\right)$.

2. $p_1 \otimes p_2 = \left(\log_r\left(1 + \frac{(r^{\mu_{p_1}} - 1)(r^{\mu_{p_2}} - 1)}{r - 1}\right), 1 - \log_r\left(1 + \frac{(r^{1-\eta_{p_1}} - 1)(r^{1-\eta_{p_2}} - 1)}{r - 1}\right), 1 - \log_r\left(1 + \frac{(r^{1-\nu_{p_1}} - 1)(r^{1-\nu_{p_2}} - 1)}{r - 1}\right)\right)$.

3. $\xi p = \left(1 - \log_r\left(1 + \frac{(r^{1-\mu_p} - 1)^{\xi}}{(r - 1)^{\xi-1}}\right), \log_r\left(1 + \frac{(r^{\eta_p} - 1)^{\xi}}{(r - 1)^{\xi-1}}\right), \log_r\left(1 + \frac{(r^{\nu_p} - 1)^{\xi}}{(r - 1)^{\xi-1}}\right)\right)$.

4. p^{ξ}=$\Bigg(\log_r\left(1+\frac{(r^{\mu_p}-1)^{\xi}}{(r-1)^{\xi-1}}\right),$
$1-\log_r\left(1+\frac{(r^{1-\eta_p}-1)^{\xi}}{(r-1)^{\xi-1}}\right),$
$1-\log_r\left(1+\frac{(r^{1-\nu_p}-1)^{\xi}}{(r-1)^{\xi-1}}\right)\Bigg).$

EXAMPLE **2.** *Let $p_1 = (0.60, 0.20, 0.08)$ and $p_2 = (0.50, 0.20, 0.15)$ be two PFNs, then by using Frank operations on PFNs as defined in Definition 3.1, for $r = 3$ and $\xi = 4$ we have*

1. $p_1 \oplus p_2$=$(0.7325, 0.0270, 0.0074)$.
2. $p_1 \otimes p_2$=$(0.2674, 0.9729, 0.9925)$.
3. $4p_1$=$(0.9947, 0.0002, 0)$.
4. ${p_1}^4$=$(0.0421, 0.7999, 0.5819)$.

THEOREM **3.1.** *Let $p = (\mu_p, \eta_p, \nu_p)$, $p_1 = (\mu_{p_1}, \eta_{p_1}, \nu_{p_1})$ and $p_2 = (\mu_{p_2}, \eta_{p_2}, \nu_{p_2})$ be any three PFNs, $r > 1$ and ξ, ξ_1, ξ_2 be any three positive real numbers, then we have*

1. $p_1 \oplus p_2 = p_2 \oplus p_1$;
2. $p_1 \otimes p_2 = p_2 \otimes p_1$;
3. $\xi(p_1 \oplus p_2) = \xi p_1 \oplus \xi p_2$;
4. $\xi_1 p \oplus \xi_2 p = (\xi_1 + \xi_2)p$;
5. $(p_1 \otimes p_2)^{\xi} = {p_1}^{\xi} \otimes {p_2}^{\xi}$;
6. $p^{\xi_1} \otimes p^{\xi_2} = p^{\xi_1+\xi_2}$.

Proof: For three PFNs $p = (\mu_p, \eta_p, \nu_p)$, $p_1 = (\mu_{p_1}, \eta_{p_1}, \nu_{p_1})$ and $p_2 = (\mu_{p_2}, \eta_{p_2}, \nu_{p_2})$ and $\xi, \xi_1, \xi_2 > 0$, according to Definition 3.1, we can obtain

1. $p_1 \oplus p_2 =$
$\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_1}}-1)(r^{1-\mu_{p_2}}-1)}{r-1}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_1}}-1)(r^{\eta_{p_2}}-1)}{r-1}\right),$
$\log_r\left(1+\frac{(r^{\nu_{p_1}}-1)(r^{\nu_{p_2}}-1)}{r-1}\right)\Bigg)$
$=\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_2}}-1)(r^{1-\mu_{p_1}}-1)}{r-1}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_2}}-1)(r^{\eta_{p_1}}-1)}{r-1}\right),$
$\log_r\left(1+\frac{(r^{\nu_{p_2}}-1)(r^{\nu_{p_1}}-1)}{r-1}\right)\Bigg) = p_2 \oplus p_1.$

2. $p_1 \otimes p_2 = \Bigg(\log_r\left(1+\frac{(r^{\mu_{p_1}}-1)(r^{\mu_{p_2}}-1)}{r-1}\right),$
$1-\log_r\left(1+\frac{(r^{1-\eta_{p_1}}-1)(r^{1-\eta_{p_2}}-1)}{r-1}\right),$
$1-\log_r\left(1+\frac{(r^{1-\nu_{p_1}}-1)(r^{1-\nu_{p_2}}-1)}{r-1}\right)\Bigg)$
$=\Bigg(\log_r\left(1+\frac{(r^{\mu_{p_2}}-1)(r^{\mu_{p_1}}-1)}{r-1}\right),$
$1-\log_r\left(1+\frac{(r^{1-\eta_{p_2}}-1)(r^{1-\eta_{p_1}}-1)}{r-1}\right),$
$1-\log_r\left(1+\frac{(r^{1-\nu_{p_2}}-1)(r^{1-\nu_{p_1}}-1)}{r-1}\right)\Bigg)$
$= p_2 \otimes p_1.$

3. $\xi(p_1 \oplus p_2)$=
$\xi\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_1}}-1)(r^{1-\mu_{p_2}}-1)}{r-1}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_1}}-1)(r^{\eta_{p_2}}-1)}{r-1}\right),$
$\log_r\left(1+\frac{(r^{\nu_{p_1}}-1)(r^{\nu_{p_2}}-1)}{r-1}\right)\Bigg)$
$=\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_1}}-1)^{\xi}(r^{1-\mu_{p_2}}-1)^{\xi}}{(r-1)^{2\xi-1}}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_1}}-1)^{\xi}(r^{\eta_{p_2}}-1)^{\xi}}{(r-1)^{2\xi-1}}\right),$
$\log_r\left(1+\frac{(r^{\nu_{p_1}}-1)^{\xi}(r^{\nu_{p_2}}-1)^{\xi}}{(r-1)^{2\xi-1}}\right)\Bigg).$
Now,
$\xi p_1 \oplus \xi p_2 = \Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_1}}-1)^{\xi}}{(r-1)^{\xi}}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_1}}-1)^{\xi}}{(r-1)^{\xi}}\right), \log_r\left(1+\frac{(r^{\nu_{p_1}}-1)^{\xi}}{(r-1)^{\xi}}\right)\Bigg) \oplus$
$\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_2}}-1)^{\xi}}{(r-1)^{\xi}}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_2}}-1)^{\xi}}{(r-1)^{\xi}}\right), \log_r\left(1+\frac{(r^{\nu_{p_2}}-1)^{\xi}}{(r-1)^{\xi}}\right)\Bigg)$
$=\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_1}}-1)^{\xi}(r^{1-\mu_{p_2}}-1)^{\xi}}{(r-1)^{2\xi-1}}\right),$
$\log_r\left(1+\frac{(r^{\eta_{p_1}}-1)^{\xi}(r^{\eta_{p_2}}-1)^{\xi}}{(r-1)^{2\xi-1}}\right),$
$\log_r\left(1+\frac{(r^{\nu_{p_1}}-1)^{\xi}(r^{\nu_{p_2}}-1)^{\xi}}{(r-1)^{2\xi-1}}\right)\Bigg).$
Therefore, $\xi(p_1 \oplus p_2) = \xi p_1 \oplus \xi p_2$.

4. $\xi_1 p \oplus \xi_2 p =$
$\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_p}-1)^{\xi_1}}{(r-1)^{\xi_1}}\right),$
$\log_r\left(1+\frac{(r^{\eta_p}-1)^{\xi_1}}{(r-1)^{\xi}}\right),$
$\log_r\left(1+\frac{(r^{\nu_p}-1)^{\xi_1}}{(r-1)^{\xi}}\right)\Bigg) \oplus$
$\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_p}-1)^{\xi_2}}{(r-1)^{\xi_2}}\right),$

$$\log_r\left(1+\frac{(r^{\eta_p}-1)^{\xi_2}}{(r-1)^{\xi_2}}\right), \log_r\left(1+\frac{(r^{\nu_p}-1)^{\xi_2}}{(r-1)^{\xi_2}}\right)\Bigg)$$
$$=\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_p}-1)^{\xi_1+\xi_2}}{(r-1)^{\xi_1+\xi_2}}\right),$$
$$\log_r\left(1+\frac{(r^{\eta_p}-1)^{\xi_1+\xi_2}}{(r-1)^{\xi_1+\xi_2}}\right),$$
$$\log_r\left(1+\frac{(r^{\nu_p}-1)^{\xi_1+\xi_2}}{(r-1)^{\xi_1+\xi_2}}\right)\Bigg)$$
$$=(\xi_1+\xi_2)p.$$

5. $(p_1 \otimes p_2)^{\xi} =$
$$\Bigg(\log_r\left(1+\frac{(r^{\mu_{p_1}}-1)(r^{\mu_{p_2}}-1)}{r-1}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\eta_{p_1}}-1)(r^{1-\eta_{p_2}}-1)}{r-1}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\nu_{p_1}}-1)(r^{1-\nu_{p_2}}-1)}{r-1}\right)\Bigg)^{\xi}$$
$$=\Bigg(\log_r\left(1+\frac{((r^{\mu_{p_1}}-1)(r^{\mu_{p_2}}-1))^{\xi}}{(r-1)^{2\xi-1}}\right),$$
$$1-\log_r\left(1+\frac{((r^{1-\eta_{p_1}}-1)(r^{1-\eta_{p_2}}-1))^{\xi}}{(r-1)^{2\xi-1}}\right),$$
$$1-\log_r\left(1+\frac{((r^{1-\nu_{p_1}}-1)(r^{1-\nu_{p_2}}-1))^{\xi}}{(r-1)^{2\xi-1}}\right)\Bigg)$$
$$=\Bigg(\log_r\left(1+\frac{(r^{\mu_{p_1}}-1)^{\xi}}{(r-1)^{\xi}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\eta_{p_1}}-1)^{\xi}}{(r-1)^{\xi}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\nu_{p_1}}-1)^{\xi}}{(r-1)^{\xi}}\right)\Bigg)\otimes$$
$$\Bigg(\log_r\left(1+\frac{(r^{\mu_{p_2}}-1)^{\xi}}{(r-1)^{\xi}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\eta_{p_2}}-1)^{\xi}}{(r-1)^{\xi}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\nu_{p_2}}-1)^{\xi}}{(r-1)^{\xi}}\right)\Bigg)$$
$$={p_1}^{\xi}\otimes{p_2}^{\xi}.$$

6. $p^{\xi_1}\otimes p^{\xi_2} =$
$$\Bigg(\log_r\left(1+\frac{(r^{\mu_p}-1)^{\xi_1}}{(r-1)^{\xi_1-1}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\eta_p}-1)^{\xi_1}}{(r-1)^{\xi_1-1}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\nu_p}-1)^{\xi_1}}{(r-1)^{\xi_1-1}}\right)\Bigg)\otimes$$
$$\Bigg(\log_r\left(1+\frac{(r^{\mu_p}-1)^{\xi_2}}{(r-1)^{\xi_2-1}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\eta_p}-1)^{\xi_2}}{(r-1)^{\xi_2-1}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\nu_p}-1)^{\xi_2}}{(r-1)^{\xi_2-1}}\right)\Bigg)$$
$$=\Bigg(\log_r\left(1+\frac{(r^{\mu_p}-1)^{\xi_1+\xi_2}}{(r-1)^{\xi_1+\xi_2-1}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\eta_p}-1)^{\xi_1+\xi_2}}{(r-1)^{\xi_1+\xi_2-1}}\right),$$
$$1-\log_r\left(1+\frac{(r^{1-\nu_p}-1)^{\xi_1+\xi_2}}{(r-1)^{\xi_1+\xi_2-1}}\right)\Bigg)$$
$$=p^{\xi_1+\xi_2}.$$ □

3.1 Picture fuzzy Frank arithmetic aggregation operators

Definition 3.2. Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})(i = 1, 2, \ldots, n)$ be a number of PFNs. Then PFFWA operator is a function $p^n \to p$ such that,

$$PFFWA(p_1, p_2, \ldots, p_n) = \bigoplus_{i=1}^{n} w_i p_i$$

where $w = (w_1, w_2, \ldots, w_n)^t$ be the weight vector of $p_i (i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Hence, we get consequential theorem that follows the Frank operations on PFNs.

THEOREM **3.2.** *Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})(i = 1, 2, \ldots, n)$ be a number of PFNs, then aggregated value of them using PFFWA operator is also a PFN, and*

$$PFFWA(p_1, p_2, \ldots, p_n) = \bigoplus_{i=1}^{n} w_i p_i$$
$$=\Bigg(1-\log_r\left(1+\prod_{i=1}^{n}(r^{1-\mu_{p_i}}-1)^{w_i}\right),$$
$$\log_r\left(1+\prod_{i=1}^{n}(r^{\eta_{p_i}}-1)^{w_i}\right),$$
$$\log_r\left(1+\prod_{i=1}^{n}(r^{\nu_{p_i}}-1)^{w_i}\right)\Bigg).$$

Proof: We prove this theorem by the method of mathematical induction.

For $n = 2$, based on Frank operations of PFNs we get the corresponding result

$$PFFWA(p_1, p_2) = \bigoplus_{i=1}^{2} w_i = w_1 p_1 \oplus w_2 p_2$$
$$=\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_1}}-1)^{w_1}}{(r-1)^{w_1-1}}\right),$$
$$\log_r\left(1+\frac{(r^{\eta_{p_1}}-1)^{w_1}}{(r-1)^{w_1-1}}\right), \log_r\left(1+\frac{(r^{\nu_{p_1}}-1)^{w_1}}{(r-1)^{w_1-1}}\right)\Bigg)$$
$$\bigoplus\Bigg(1-\log_r\left(1+\frac{(r^{1-\mu_{p_2}}-1)^{w_2}}{(r-1)^{w_2-1}}\right),$$
$$\log_r\left(1+\frac{(r^{\eta_{p_2}}-1)^{w_2}}{(r-1)^{w_2-1}}\right), \log_r\left(1+\frac{(r^{\nu_{p_2}}-1)^{w_2}}{(r-1)^{w_2-1}}\right)\Bigg)$$

$$= \Bigg(1-\log_r\Big(1+\prod_{i=1}^{2}(r^{1-\mu_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{2}(r^{\eta_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{2}(r^{\nu_{p_i}}-1)^{w_i}\Big)\Bigg)[\because \sum_{i=1}^{2}w_i=1]$$

Hence, the result is valid for $n=2$.

Let us assume that, the given result is true for $n=s$. Therefore, we have,

$$PFFWA(p_1,p_2,\ldots,p_s)=\bigoplus_{i=1}^{s}w_ip_i$$
$$= \Bigg(1-\log_r\Big(1+\prod_{i=1}^{s}(r^{1-\mu_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{s}(r^{\eta_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{s}(r^{\nu_{p_i}}-1)^{w_i}\Big)\Bigg)$$

Now, for $n=s+1$

$$PFFWA(p_1,p_2,\ldots,p_s,p_{s+1})=$$
$$\bigoplus_{i=1}^{s+1}w_ip_i=\bigoplus_{i=1}^{s}w_ip_i\bigoplus w_{s+1}p_{s+1}$$
$$= \Bigg(1-\log_r\Bigg(1+\frac{\prod_{i=1}^{s}(r^{1-\mu_{p_i}}-1)^{w_i}}{(r-1)^{\sum_{i=1}^{s}w_i-1}}\Bigg),$$
$$\log_r\Bigg(1+\frac{\prod_{i=1}^{s}(r^{\eta_{p_i}}-1)^{w_i}}{(r-1)^{\sum_{i=1}^{s}w_i-1}}\Bigg),$$
$$\log_r\Bigg(1+\frac{\prod_{i=1}^{s}(r^{\nu_{p_i}}-1)^{w_i}}{(r-1)^{\sum_{i=1}^{s}w_i-1}}\Bigg)\Bigg)\bigoplus$$
$$\Bigg(1-\log_r\Big(1+\frac{(r^{1-\mu_{p_{s+1}}}-1)^{w_{s+1}}}{(r-1)^{w_{s+1}-1}}\Big),$$
$$\log_r\Big(1+\frac{(r^{\eta_{p_{s+1}}}-1)^{w_{s+1}}}{(r-1)^{w_{s+1}-1}}\Big),$$
$$\log_r\Big(1+\frac{(r^{\nu_{p_{s+1}}}-1)^{w_{s+1}}}{(r-1)^{w_{s+1}-1}}\Big)\Bigg)$$
$$= \Bigg(1-\log_r\Big(1+\prod_{i=1}^{s+1}(r^{1-\mu_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{s+1}(r^{\eta_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{s+1}(r^{\nu_{p_i}}-1)^{w_i}\Big)\Bigg)[\text{as} \sum_{i=1}^{s+1}w_i=1]$$

Therefore, the result is true for $n=s+1$ if it is true for $n=s$. Also it is true for $n=2$. Hence, by the method of induction the given result is true for any natural number n. □

THEOREM **3.3.** *(Idempotency Property). If $p_i=(\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of identical PFNs, i.e., $p_i=p$ for all i, where $p=(\mu_p,\eta_p,\nu_p)$, then $PFFWA(p_1,p_2,\ldots,p_n)=p$.*

Proof: Since $p_i=p$ for all i then, we have

$$PFFWA(p_1,p_2,\ldots,p_n)$$
$$= \Bigg(1-\log_r\Big(1+\prod_{i=1}^{n}(r^{1-\mu_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{n}(r^{\eta_{p_i}}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{n}(r^{\nu_{p_i}}-1)^{w_i}\Big)\Bigg)$$
$$= \Bigg(1-\log_r\Big(1+\prod_{i=1}^{n}(r^{1-\mu_p}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{n}(r^{\eta_p}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{n}(r^{\nu_p}-1)^{w_i}\Big)\Bigg)$$
$$= \Bigg(1-\log_r\Big(1+(r^{1-\mu_p}-1)^{\sum_{i=1}^{n}w_i}\Big),$$
$$\log_r\Big(1+(r^{\eta_p}-1)^{\sum_{i=1}^{n}w_i}\Big),$$
$$\log_r\Big(1+(r^{\nu_p}-1)^{\sum_{i=1}^{n}w_i}\Big)\Bigg)$$
$$= \Bigg(1-\log_r\Big(1+(r^{1-\mu_p}-1)\Big),$$
$$\log_r\Big(1+(r^{\eta_p}-1)\Big),$$
$$\log_r\Big(1+(r^{\nu_p}-1)\Big)\Bigg)=\Big(\mu_p,\eta_p,\nu_p\Big)=p.$$

Hence the result follows. □

THEOREM **3.4.** *(Boundedness property). Let $p_i=(\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs. Let $p^-=min\{p_1,p_2,\ldots,p_n\}$ and $p^+=max\{p_1,p_2,\ldots,p_n\}$. Then, $p^-\leq PFFWA(p_1,p_2,\ldots,p_n)\leq p^+$.*

Proof: Let $p_i=(\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs. Let $p^-=min\{p_1,p_2,\ldots,p_n\}=(\mu^-,\eta^-,\nu^-)$ and $p^+=max\{p_1,p_2,\ldots,p_n\}=(\mu^+,\eta^+,\nu^+)$. We have $\mu^-=\min_k\{\mu_{p_k}\}$,$\eta^-=\max_k\{\eta_{p_k}\}$,$\nu^-=\max_k\{\nu_{p_k}\}$, $\mu^+=\max_k\{\mu_{p_k}\}$, $\eta^+=\min_k\{\eta_{p_k}\}$ and $\nu^+=\min_k\{\nu_{p_k}\}$.

Now,

$$1-\log_r\Big(1+\prod_{i=1}^{n}(r^{1-(\mu^-)}-1)^{w_i}\Big)\leq$$
$$1-\log_r\Big(1+\prod_{i=1}^{n}(r^{1-\mu_{p_i}}-1)^{w_i}\Big)\leq$$
$$1-\log_r\Big(1+\prod_{i=1}^{n}(r^{1-(\mu^+)}-1)^{w_i}\Big),$$
$$\log_r\Big(1+\prod_{i=1}^{n}(r^{(\eta^+)}-1)^{w_i}\Big)\Big)\leq$$
$$\log_r\Big(1+\prod_{i=1}^{n}(r^{\eta_{p_i}}-1)^{w_i}\Big)\Big)\leq$$

$\log_r\left(1+\prod_{i=1}^{n}(r^{(\eta^-)}-1)^{w_i}\right)\Big)$,
$\log_r\left(1+\prod_{i=1}^{n}(r^{(\nu^+)}-1)^{w_i}\right)\Big) \leq$
$\log_r\left(1+\prod_{i=1}^{n}(r^{\nu_{p_i}}-1)^{w_i}\right)\Big) \leq$
$\log_r\left(1+\prod_{i=1}^{n}(r^{(\nu^-)}-1)^{w_i}\right)\Big)$.
Therefore, $p^- \leq PFFWA(p_1,p_2,\ldots,p_n) \leq p^+$. □

THEOREM **3.5.** *(Monotonicity property) Let p_i and p'_i(i=1, 2,..., n) be two sets of PFNs, if $p_i \leq p'_i$ for all i, then $PFFWA(p_1,p_2,\ldots,p_n) \leq PFFWA(p'_1,p'_2,\ldots,p'_n)$.*

Proof: Since $p_i \leq p'_i$ for all $i = 1,2,...,n$, then, we have $\mu_{p_i} \leq \mu'_{p_i}$, $\eta_{p_i} \leq \eta'_{p_i}$ and $\nu_{p_i} \geq \nu'_{p_i}$ for all $i = 1,2,\ldots,n$.
Now, $(r^{1-\mu_{p_i}}-1)^{w_i} \geq (r^{1-\mu'_{p_i}}-1)^{w_i}$
$\Rightarrow \log_r\left(1+\prod_{i=1}^{n}(r^{1-\mu_{p_i}}-1)^{w_i}\right) \geq$
$\log_r\left(1+\prod_{i=1}^{n}(r^{1-\mu'_{p_i}}-1)^{w_i}\right)$
$\Rightarrow 1-\log_r\left(1+\prod_{i=1}^{n}(r^{1-\mu_{p_i}}-1)^{w_i}\right) \leq$
$1-\log_r\left(1+\prod_{i=1}^{n}(r^{1-\mu'_{p_i}}-1)^{w_i}\right)$.
Similarly, it can be shown that
$\log_r\left(1+\prod_{i=1}^{n}(r^{\eta_{p_i}}-1)^{w_i}\right) \leq$
$\log_r\left(1+\prod_{i=1}^{n}(r^{\eta'_{p_i}}-1)^{w_i}\right)$
and
$\log_r\left(1+\prod_{i=1}^{n}(r^{\nu_{p_i}}-1)^{w_i}\right) \geq$
$\log_r\left(1+\prod_{i=1}^{n}(r^{\nu'_{p_i}}-1)^{w_i}\right)$.
Therefore, $PFFWA(p_1,p_2,\ldots,p_n) \leq$
$PFFWA(p'_1,p'_2,\ldots,p'_n)$. □

Now, we would like to introduce PFFOWA operator.

Definition 3.3. Let $p_i = (\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs. The PFFOWA operator of dimension n is a function $p^n \to p$ such that,

$$PFFOWA(p_1,p_2,\ldots p_n) = \bigoplus_{i=1}^{n} w_i p_{\rho(i)}$$

where $w = (w_1,w_2,\ldots,w_n)^t$ be the weight vector of $p_i(i=1,2,\ldots,n)$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, $(\rho(1),\rho(2),\ldots,\rho(n))$ is the permutation of $(i = 1,2,\ldots,n)$, for which $p_{\rho(i-1)} \geq p_{\rho(i)}$ for all $i = 1,2,\ldots,n$.

Based on Frank product of PFNs the following theorem is developed.

THEOREM **3.6.** *Let $p_i = (\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs. The PFFOWA operator of dimension n is a function $p^n \to p$ with the corresponding weight vector $w = (w_1,w_2,\ldots,w_n)^t$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then,*

$PFFOWA(p_1,p_2,\ldots,p_n) = \bigoplus_{i=1}^{n} w_i p_{\rho(i)}$
$= \Bigg(1-\log_r\left(1+\prod_{i=1}^{n}(r^{1-\mu_{p_{\rho(i)}}}-1)^{w_i}\right)$,
$\log_r\left(1+\prod_{i=1}^{n}(r^{\eta_{p_{\rho(i)}}}-1)^{w_i}\right)$,
$\log_r\left(1+\prod_{i=1}^{n}(r^{\nu_{p_{\rho(i)}}}-1)^{w_i}\right)\Bigg)$
where $(\rho(1),\rho(2),\ldots,\rho(n))$ are the permutation of $(i = 1,2,\ldots,n)$ for which $p_{\rho(i-1)} \geq p_{\rho(i)}$ for all $i = 1,2,\ldots,n$.

With the help of PFFOWA operator we can easily prove the following properties.

THEOREM **3.7.** *(Idempotency property). If $p_i = (\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs all are identical, i.e., $p_i = p$ for all i. Then, $PFFOWA(p_1,p_2,\ldots,p_n) = p$.*

THEOREM **3.8.** *(Boundedness Property). Let $p_i = (\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs. Let $p^- = min\{p_1,p_2,\ldots,p_n\}$ and $p^+ = max\{p_1,p_2,\ldots,p_n\}$. Then, $p^- \leq PFFOWA(p_1,p_2,\ldots,p_n) \leq p^+$.*

THEOREM **3.9.** *(Monotonicity Property). Let p_i and p'_i(i=1, 2,..., n) be two sets of PFNs, if $p_i \leq p'_i$ for all i, then $PFFOWA(p_1,p_2,\ldots,p_n) \leq PFFOWA(p'_1,p'_2,\ldots,p'_n)$.*

THEOREM **3.10.** *(Commutative Property). Let p_i and p'_i(i=1, 2,..., n) be two sets of PFNs, then $PFFOWA(p_1,p_2,\ldots,p_n) = PFFOWA(p'_1,p'_2,\ldots,p'_n)$ where p'_i is any permutation of $p_i(i=1,2,\ldots,n)$.*

In Definition 3.2, we find that the weights associated with the PFFWA operator are the simplest form of PF value and in Definition 3.3 the weights associated with the PF-FOWA operator is the original form of the ordered positions of the PF values. In this way, the weights disclosed in the PFFWA and PFFOWA operators, present various perspectives which are conflicting with one another. But, these perspectives are deliberated to be the same in a general approach. Only to be rescued of such drawback, we now introduce PFFHA operator.

Definition 3.4. Let $p_i = (\mu_{p_i},\eta_{p_i},\nu_{p_i})(i=1,2,\ldots,n)$ be a number of PFNs. The PFFHA operator of dimension

n is a function $p^n \to p$ such that,

$$
\begin{aligned}
&PFFHA(p_1, p_2, \ldots, p_n) \\
=\ & \bigoplus_{i=1}^{n} \bar{w}_i \dot{p}_{\rho(i)} \\
=\ & \Bigg(1 - \log_r\Big(1 + \prod_{i=1}^{n}(r^{1-\dot{\mu}_{p_{\rho(i)}}} - 1)^{\bar{w}_i}\Big), \\
& \log_r\Big(1 + \prod_{i=1}^{n}(r^{\dot{\eta}_{p_{\rho(i)}}} - 1)^{\bar{w}_i}\Big), \\
& \log_r\Big(1 + \prod_{i=1}^{n}(r^{\dot{\nu}_{p_{\rho(i)}}} - 1)^{\bar{w}_i}\Big)\Bigg)
\end{aligned}
$$

where $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^t$ is the aggregation associated weight vector, $\sum_{i=1}^{n} \bar{w}_i = 1$, $w = (w_1, w_2, \ldots, w_n)^t$ be the weight vector of $p_i (i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. $\dot{p}_{\rho(i)}$ is the i^{th} biggest weighted PF values of $\dot{p}_i (\dot{p}_i = n w_i p_i, i = 1, 2, \ldots, n)$, n is the balancing coefficient.

Deduction 3.1. *When $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t$, then $\dot{p}_i = n \times \frac{1}{n} \times p_i = p_i$ for $i = 1, 2, ..., n$. Then the PFFHA operator degenerates into PFFOWA operator. If $\bar{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t$, then PFFHA operator reduces to PFFWA operator. Hence, PFFWA and PFFOWA operators are a specific type of PFFHA operator. Thus, PFFHA operator is a generalization of both the PFFWA and PFFOWA operators, which reflects the degrees of the stated disagreements and their organized situations.*

3.2 Picture fuzzy Frank geometric aggregation operators

Definition 3.5. Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n)$ be a number of PFNs. Then PFFWG operator is a function $p^n \to p$ such that,

$$PFFWG(p_1, p_2, \ldots, p_n) = \bigotimes_{i=1}^{n} (p_i)^{w_i}$$

where $w = (w_1, w_2, \ldots, w_n)^t$ be the weight vector of $p_i (i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Hence, we get consequential theorem that follows the Frank operations on PFNs.

THEOREM **3.11.** *Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n)$ be a number of PFNs, then aggregated value of them using PFFWG operator is also a PFN, and*

$$
\begin{aligned}
&PFFWG(p_1, p_2, \ldots, p_n) = \bigotimes_{i=1}^{n} (p_i)^{w_i} \\
&= \Bigg(\log_r\Big(1 + \prod_{i=1}^{n}(r^{\mu_{p_i}} - 1)^{w_i}\Big), \\
&1 - \log_r\Big(1 + \prod_{i=1}^{n}(r^{1-\eta_{p_i}} - 1)^{w_i}\Big), \\
&1 - \log_r\Big(1 + \prod_{i=1}^{n}(r^{1-\nu_{p_i}} - 1)^{w_i}\Big)\Bigg).
\end{aligned}
$$

Proof: The proof of this theorem emulates from Theorem 3.2. □

The following properties may be easily proved by PFFWG operator.

THEOREM **3.12.** *(Idempotency Property). If $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n)$ be a number of identical PFNs, i.e., $p_i = p$ for all i. Then, $PFFWG(p_1, p_2, \ldots, p_n) = p$.*

THEOREM **3.13.** *(Boundedness Property). Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n)$ be a number of PFNs. Let $p^- = min\{p_1, p_2, \ldots, p_n\}$ and $p^+ = max\{p_1, p_2, \ldots, p_n\}$. Then, $p^- \leq PFFWG(p_1, p_2, \ldots, p_n) \leq p^+$.*

THEOREM **3.14.** *(Monotonicity Property). Let p_i and p_i'(i=1, 2,..., n) be two sets of PFNs, if $p_i \leq p_i'$ for all i, then $PFFWG(p_1, p_2, \ldots, p_n) \leq PFFWG(p_1', p_2', \ldots, p_n')$.*

Now, we would like to introduce PFFOWG operator.

Definition 3.6. Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n)$ be a number of PFNs. The PFFOWG operator of dimension n is a function $p^n \to p$ such that,

$$PFFOWG(p_1, p_2, \ldots p_n) = \bigotimes_{i=1}^{n} (p_{\rho(i)})^{w_i}$$

where $w = (w_1, w_2, \ldots, w_n)^t$ be the weight vector of $p_i (i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, $(\rho(1), \rho(2), \ldots, \rho(n))$ are the permutation of $(i = 1, 2, \ldots, n)$, for which $p_{\rho(i-1)} \geq p_{\rho(i)}$ for all $i = 1, 2, \ldots, n$.

The following theorem is developed based on Frank product operation on PFNs using PFFOWG operator.

THEOREM **3.15.** *Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n)$ be a number of PFNs. The PFFOWG operator of dimension n is a function $p^n \to p$ with the corresponding weight vector $w = (w_1, w_2, \ldots, w_n)^t$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then,*

$$
\begin{aligned}
&PFFOWG(p_1, p_2, \ldots p_n) = \bigotimes_{i=1}^{n} (p_{\rho(i)})^{w_i} \\
&= \Bigg(\log_r\Big(1 + \prod_{i=1}^{n}(r^{\mu_{p_{\rho(i)}}} - 1)^{w_i}\Big), \\
&1 - \log_r\Big(1 + \prod_{i=1}^{n}(r^{1-\eta_{p_{\rho(i)}}} - 1)^{w_i}\Big), \\
&1 - \log_r\Big(1 + \prod_{i=1}^{n}(r^{1-\nu_{p_{\rho(i)}}} - 1)^{w_i}\Big)\Bigg)
\end{aligned}
$$

where $(\rho(1), \rho(2), \ldots, \rho(n))$ are the permutation of $(i = 1, 2, \ldots, n)$ for which $p_{\rho(i-1)} \geq p_{\rho(i)}$ for all $i = 1, 2, \ldots, n$.

The following properties can be investigated by using PFFOWG operator.

THEOREM **3.16.** *(Idempotency property). If* $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})(i = 1, 2, \ldots, n)$ *be a number of PFNs all are identical, i.e.,* $p_i = p$ *for all i. Then,* $PFFOWG(p_1, p_2, \ldots, p_n) = p$.

THEOREM **3.17.** *(Boundedness Property). Let* $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})(i = 1, 2, \ldots, n)$ *be a number of PFNs. Let* $p^- = min\{p_1, p_2, \ldots, p_n\}$ *and* $p^+ = max\{p_1, p_2, \ldots, p_n\}$. *Then,* $p^- \leq PFFOWG(p_1, p_2, \ldots, p_n) \leq p^+$.

THEOREM **3.18.** *(Monotonicity Property). Let* p_i *and* p'_i*(i=1, 2,..., n) be two sets of PFNs, if* $p_i \leq p'_i$ *for all i, then* $PFFOWG(p_1, p_2, \ldots, p_n) \leq PFFOWG(p'_1, p'_2, \ldots, p'_n)$.

THEOREM **3.19.** *(Commutative Property). Let* p_i *and* p'_i*(i=1, 2,..., n) be two sets of PFNs, then* $PFFOWG(p_1, p_2, \ldots, p_n) = PFFOWG(p'_1, p'_2, \ldots, p'_n)$ *where* p'_i *is any permutation of* $p_i(i = 1, 2, \ldots, n)$.

In Definition 3.5, we find that the weights associated with the PFFWG operator are in the simplest form of PF value and in Definition 3.6 the weights associated with the PFFOWG operator are in the actual form of the ordered positions of the PF values. In this way, the weights disclosed in the PFFWG and PFFOWG operators, present various perspectives which are conflicting with one another. But, these perspectives are deliberated to be the same in a general approach. Only to be rescued of such drawback, we at this moment introduce PFFHG operator.

Definition 3.7. Let $p_i = (\mu_{p_i}, \eta_{p_i}, \nu_{p_i})(i = 1, 2, \ldots, n)$ be a number of PFNs. The PFFHG operator of dimension n is a function $p^n \to p$ such that,

$$PFFHG(p_1, p_2, \ldots, p_n) = \bigotimes_{i=1}^{n} (\dot{p}_{\rho(i)})^{\bar{w}_i}$$
$$= \Bigg(\log_r \Big(1 + \prod_{i=1}^{n} (r^{\dot{\mu}_{p_{\rho(i)}}} - 1)^{\bar{w}_i}\Big),$$
$$1 - \log_r \Big(1 + \prod_{i=1}^{n} (r^{1-\dot{\eta}_{p_{\rho(i)}}} - 1)^{\bar{w}_i}\Big),$$
$$1 - \log_r \Big(1 + \prod_{i=1}^{n} (r^{1-\dot{\nu}_{p_{\rho(i)}}} - 1)^{\bar{w}_i}\Big)\Bigg)$$

where $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^t$ is the aggregation associated weight vector, $\sum_{i=1}^{n} \bar{w}_i = 1$, $w = (w_1, w_2, \ldots, w_n)^t$ be the weight vector of $p_i(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. $\dot{p}_{\rho(i)}$ is the i^{th} biggest weighted PF values of $\dot{p}_i(\dot{p}_i = nw_ip_i, i = 1, 2, \ldots, n)$, n is the balancing coefficient.

Deduction 3.2. *When* $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t$, *then* $\dot{p}_i = n \times \frac{1}{n} \times p_i = p_i$ *for* $i = 1, 2, \ldots, n$. *Then the PFFHG operator degenerates into PFFOWG operator. If* $\bar{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^t$, *then PFFHG operator reduces to PFFWG operator. Hence, PFFWG and PFFOWG operators are specific types of PFFHG operator. Thus, PFFHG operator is a generalization of both the PFFWG and PFFOWG operators, which reflects the degrees of the stated disagreements and their organized situations.*

4 Model for MADM using picture fuzzy data

In this section, we introduce a novel approach for decision-making problems using PF information manipulating PFFWA and PFFWG operators, where attribute values are PFNs and attribute weights are real numbers. For an MADM problem, let $F = \{F_1, F_2, \ldots, F_m\}$ be a discrete set of m alternatives to be selected and $H = \{H_1, H_2, \ldots, H_n\}$ be the arrangement of attributes to be assessed. Let $w = \{w_1, w_2, \ldots, w_n\}$ be the weight vector of the attributes $H_j(j = 1, 2, \ldots, n)$ where $w_k(k = 1, 2, 3, \ldots, n)$ are all real numbers such that $w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$. Assume that $P = (\gamma_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, \nu_{ij}))_{m\times n}$ is the PF decision matrix, where γ_{ij} is the possible value for which the alternative F_i satisfies the attribute H_j where $\mu_{ij} + \eta_{ij} + \nu_{ij} \leq 1$ and $\mu_{ij},\ \eta_{ij},\ \nu_{ij} \in [0, 1]$.

To achieve the final ranking of the alternatives, we propose an algorithm which is shown in the following.

4.1 Algorithm

The proposed MADM problem with PF data based on the proposed PFFWA and PFFWG operators is now presented as follows:

Step I: Construct the PF decision matrix $P = (\gamma_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, \nu_{ij}))_{m\times n}$.

Step II: Transform the matrix $P = (\gamma_{ij})_{m\times n} = ((\mu_{ij}, \eta_{ij}, \nu_{ij}))_{m\times n}$ into a normalize PF matrix $P' = (\gamma'_{ij})_{m\times n} = ((\mu'_{ij}, \eta'_{ij}, \nu'_{ij}))_{m\times n}$ by Equation (1).

$$\gamma'_{ij} = \begin{cases} (\mu_{ij}, \eta_{ij}, \nu_{ij}), & \text{if } H_j \text{ is benefit attribute;} \\ (\nu_{ij}, \eta_{ij}, \mu_{ij}), & \text{if } H_j \text{ is cost attribute.} \end{cases} \tag{1}$$

Step III: Calculate the collective information σ_k of the alternative F_k by Equations (2) and (3).

$$\begin{aligned} \sigma_k &= PFFWA(\gamma'_{k1}, \gamma'_{k2}, \ldots, \gamma'_{kn}) \\ &= \bigoplus_{j=1}^{n} (w_j \gamma_{kj}) \\ &= \Bigg(1 - \log_r \Big(1 + \prod_{i=1}^{n} (r^{1-\mu'_{p_{kj}}} - 1)^{w_j}\Big), \\ &\quad \log_r \Big(1 + \prod_{i=1}^{n} (r^{\eta'_{p_{kj}}} - 1)^{w_j}\Big), \\ &\quad \log_r \Big(1 + \prod_{i=1}^{n} (r^{\nu'_{p_{kj}}} - 1)^{w_j}\Big)\Bigg). \end{aligned} \tag{2}$$

and

$$\begin{aligned}\sigma_k &= PFFWG(\gamma'_{k1}, \gamma'_{k2}, \dots, \gamma'_{kn}) \\ &= \bigotimes_{j=1}^{n} (\gamma_{kj})^{w_j} \\ &= \Bigg(\log_r \Big(1 + \prod_{i=1}^{n} (r^{\mu'_{p_{kj}}} - 1)^{w_j} \Big), \\ &\quad 1 - \log_r \Big(1 + \prod_{i=1}^{n} (r^{1-\eta'_{p_{kj}}} - 1)^{w_j} \Big), \\ &\quad 1 - \log_r \Big(1 + \prod_{i=1}^{n} (r^{1-\nu'_{p_{kj}}} - 1)^{w_j} \Big) \Bigg). \quad (3)\end{aligned}$$

Step IV: Compute the score function $\Delta(\sigma_i)$ for each alternative using Definition 2.3.

Step V: The optimal decision is to select F_k if $\Delta(\sigma_k) = \max_{l}\{\Delta(\sigma_l)\}$.

5 Numerical illustration

In this section, we are willing to sketch a numerical problem to illustrate the possible assessment of commercialization with the help of PF data.

Suppose a renowned multi-tasking company has decided to utilize a part of its total annual profit in some improvement of the company's good-will. The managing board has selected some alternative choices to invest the fund, such as

1. F_1 : Air conditioning and furnishing the whole floor.
2. F_2 : Purchasing of some advanced gadgets.
3. F_3 : Constructing a parking zone.
4. F_4 : Advertising.
5. F_5 : Security facility.

Now, since each alternative satisfies different requirements so, confusion arises to make a decision. Thereby, the managing board has determined the following considerable attributes,

- H_1 : Enhancement of profit.
- H_2 : Customer's benefit.
- H_3 : Maintenance cost.
- H_4 : Ecofriendliness.

Now the decision making in this case is difficult because each alternative promises to maximize a different attribute. The managing board defines the weight vector of the attribute $H_j (j = 1, 2, 3, 4)$ as $(0.30, 0.25, 0.20, 0.25)$. Meanwhile, H_1, H_2, H_4 are benefit attributes and H_3 is a cost attribute. Assume that the alternative F_i with respect to the attribute H_j is expressed as PF matrix $P = (\gamma_{ij})_{m\times n} = ((\mu_{ij}, \nu_{ij}))_{m\times n}$. The assessment for the alternatives are shown in the Table 1.

In order to select the most preferable alternative $F_i(i = 1, 2, 3, 4, 5)$ we exploit the PFFWA and PFFWG operators to develop an MADM theory with PF data, which can be evaluated as follows:

Step 1: We input the PF decision matrix given in Table 1.

Step 2: By normalizing of PF decision matrix with the help of Equation (1) we get the matrix N.

Step 3: We take $r = 2$ and use PFFWA operator to compute overall performance values $\sigma_i(i = 1, 2, 3, 4, 5)$ of alternatives F_i's using Equation (2)

- $\sigma_1 = (0.6188, 0.1800, 0.0879)$
- $\sigma_2 = (0.6517, 0.1827, 0.1214)$
- $\sigma_3 = (0.5441, 0.2861, 0.0559)$
- $\sigma_4 = (0.6006, 0.2100, 0.0713)$
- $\sigma_5 = (0.5823, 0.1456, 0.1478)$.

Step 4: We compute the values of the score functions using Definition 2.3, $\Delta(\sigma_i)(k = 1, 2, 3, 4, 5)$ of the overall PFNs $\sigma_i(i = 1, 2, 3, 4, 5)$ as

- $\Delta(\sigma_1) = 0.7654$
- $\Delta(\sigma_2) = 0.7651$
- $\Delta(\sigma_3) = 0.7441$
- $\Delta(\sigma_4) = 0.7646$
- $\Delta(\sigma_5) = 0.7172$.

Therefore, with respect to the score values, we rank all the alternatives as $F_1 > F_2 > F_4 > F_3 > F_5$.

Step 5: Therefore, F_1 should be selected as the most preferable alternative by the company.

Again, if PFFWG operator is used instead of PFFWA operator, then the problem can be solved similarly as above.

Step 1: We input the PF decision matrix given in Table 1.

Step 2: The normalized matrix is same as the matrix N.

Step 3: We take $r = 2$ and use PFFWG operator to compute overall performance values $\sigma_i(i = 1, 2, 3, 4, 5)$ by Equation 3 of the alternatives F_i's.

- $\sigma_1 = (0.4210, 0.2959, 0.1067)$
- $\sigma_2 = (0.5968, 0.2118, 0.1277)$
- $\sigma_3 = (0.2922, 0.4242, 0.0580)$
- $\sigma_4 = (0.3057, 0.3727, 0.0778)$
- $\sigma_5 = (0.4545, 0.2031, 0.1600)$.

Step 4: We compute the values of the score function using Definition 2.3, $\Delta(\sigma_i)(i = 1, 2, 3, 4, 5)$ of the overall PFNs $\sigma_i(i = 1, 2, 3, 4, 5)$ as

- $\Delta(\sigma_1) = 0.6571$
- $\Delta(\sigma_2) = 0.7345$
- $\Delta(\sigma_3) = 0.6170$
- $\Delta(\sigma_4) = 0.6139$
- $\Delta(\sigma_5) = 0.6472$.

	H_1	H_2	H_3	H_4
F_1	(0.60,0.25,0.12)	(0.91,0.03,0.05)	(0.22,0.20,0.38)	(0.12,0.59,0.05)
F_2	(0.72,0.15,0.10)	(0.32,0.40,0.20)	(0.11,0.15,0.70)	(0.75,0.12,0.10)
F_3	(0.80,0.10,0.04)	(0.09,0.70,0.05)	(0.08,0.60,0.07)	(0.70,0.20,0.07)
F_4	(0.85,0.05,0.04)	(0.76,0.15,0.07)	(0.09,0.70,0.07)	(0.09,0.53,0.12)
F_5	(0.71,0.10,0.11)	(0.56,0.20,0.19)	(0.09,0.50,0.09)	(0.69,0.03,0.24)

Table 1: Picture fuzzy decision matrix

Therefore, with respect to the score values, we rank all the alternatives as $F_2 > F_1 > F_5 > F_3 > F_4$.

Step 5: Therefore, F_2 should be selected as the most preferable alternative by the company.

As we have demonstrated above, the score values of the alternatives are different from each other. But the ranking orders corresponding to various alternatives are the same, and the preferable alternative is always F_2.

$$N = \begin{pmatrix} (0.60, 0.25, 0.12) & (0.91, 0.03, 0.05) & (0.38, 0.20, 0.22) & (0.12, 0.59, 0.05) \\ (0.72, 0.15, 0.10) & (0.32, 0.40, 0.20) & (0.70, 0.15, 0.11) & (0.75, 0.12, 0.10) \\ (0.80, 0.10, 0.04) & (0.09, 0.70, 0.05) & (0.07, 0.60, 0.08) & (0.70, 0.20, 0.07) \\ (0.85, 0.05, 0.04) & (0.76, 0.15, 0.07) & (0.07, 0.70, 0.09) & (0.09, 0.53, 0.12) \\ (0.71, 0.10, 0.11) & (0.56, 0.20, 0.19) & (0.09, 0.50, 0.09) & (0.69, 0.03, 0.24) \end{pmatrix}$$

Next, we will show how the parameter r affects the ranking results obtained by utilizing PFFWA and PFFWG operators.

6 Analysis of the effect of the parameter r on decision making

We can utilize different values of the operational parameter r, for ranking the given alternatives in our proposed method.

For exploring the flexibility and sensitivity of the parameter r, we fix different values of r to categorize the novel numerical MADM example. Depending on PFFWA operator and PFFWG operator, the consequences of ranking orders of the alternatives F_1, F_2, F_3, F_4, F_5 for different values of the parameter r are shown in the Table 2 and Table 3.

To provide a better view of the aggregation results, we show the results of the rankings of the alternatives by the proposed PFFWA and PFFWG operators in Figure 1(a) and Figure 1(b) respectively.

From Table 2 and Figure 1(a) we can easily see that when $3 \leq r \leq 10, r = 15, 20, 25, 50$ the aggregation score values using PFFWA operator with different parameter r are different, but the ranking orders of the alternatives $F_i (i = 1, 2, 3, 4, 5)$ are same and the ranking order is $F_2 > F_1 > F_4 > F_3 > F_5$. However, when $r = 2$, we obtain $F_1 > F_2 > F_4 > F_3 > F_5$ and in that case the optimal alternative is F_1.

From Table 3 and Figure 1(b), we can see that the aggregation score values using PFFWG operator with different parameter r are different, but the optimal alternative is always F_2. When $2 \leq r \leq 9$, we obtained $F_2 > F_1 > F_5 > F_3 > F_4$, when $r = 10$, we get $F_2 > F_1 > F_5 > F_3 \sim F_4$ and for $r = 15, 20, 25, 50$, we obtained $F_2 > F_1 > F_5 > F_4 > F_3$. Hence, the overall best alternative is F_2.

In general, different decision-makers can set different values of the parameter r based on their preferences.

In this MADM problem based on PFFWA and PFFWG operators, we can notice that for PFFWG operator the ranking orders of the alternatives can be changed by the variation of values of the parameter r. Therefore, the PFFWG operator has responded more to r in this MADM method. At the same time, in correspondence with PFFWA operator according to different values of working parameter r, ranking forms can be changed. So PFFWA operator is less responsive to r in this case of the MADM procedure.

7 Comparison analysis

In order to verify the utility of the proposed method and to pursue its advantages, we compare our proposed Frank aggregation operators with other existing well-known aggregation operators under the PF environment. The comparative results are shown in Table 4. We compare our proposed method with PFWA operator [37] and PFWG operator [37].

Making a comparison with PFWA or PFWG operators introduced by Wei [37], we can find that PFWA or PFWG operator is only a particular case of our proposed operators when the parameter $r \to 1$. Therefore, indeed, our introduced procedures are more generalized. Moreover, our proposed operators, based on Frank t-norm and Frank t-conorm are more nourished and can adopt the relationship between various arguments. Also, our proposed operators present the Lukasiewicz product and Lukasiewicz sum when the parameter $r \to \infty$. Therefore, we have ar-

r	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Ranking order	Optimal alternative
2	0.7654	0.7651	0.7441	0.7646	0.7172	$F_1 > F_2 > F_4 > F_3 > F_5$	F_1
3	0.7610	0.7640	0.7396	0.7594	0.7153	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
4	0.7581	0.7632	0.7365	0.7559	0.7140	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
5	0.7558	0.7626	0.7342	0.7532	0.7130	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
6	0.7541	0.7622	0.7324	0.7511	0.7122	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
7	0.7526	0.7618	0.7309	0.7493	0.7115	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
8	0.7514	0.7614	0.7297	0.7478	0.7110	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
9	0.7503	0.7611	0.7286	0.7465	0.7105	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
10	0.7494	0.7609	0.7277	0.7454	0.7101	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
15	0.7459	0.7599	0.7243	0.7412	0.7085	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
20	0.7436	0.7593	0.7221	0.7385	0.7075	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
25	0.7419	0.7588	0.7204	0.7364	0.7068	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2
50	0.7370	0.7575	0.7160	0.7308	0.7046	$F_2 > F_1 > F_4 > F_3 > F_5$	F_2

Table 2: Effect of the parameter r on decision making result using PFFWA operator

r	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Ranking order	Optimal alternative
2	0.6571	0.7345	0.6170	0.6139	0.6472	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
3	0.6613	0.7361	0.6220	0.6197	0.6514	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
4	0.6641	0.7372	0.6255	0.6238	0.6541	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
5	0.6663	0.7380	0.6282	0.6269	0.6562	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
6	0.6679	0.7386	0.6304	0.6294	0.6577	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
7	0.6693	0.7391	0.6322	0.6315	0.6590	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
8	0.6704	0.7395	0.6337	0.6333	0.6601	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
9	0.6714	0.7398	0.6351	0.6349	0.6611	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
10	0.6732	0.7401	0.6363	0.6363	0.6619	$F_2 > F_1 > F_5 > F_3 \sim F_4$	F_2
15	0.6755	0.7412	0.6408	0.6415	0.6648	$F_2 > F_1 > F_5 > F_4 > F_3$	F_2
20	0.6776	0.7419	0.6438	0.6450	0.6668	$F_2 > F_1 > F_5 > F_4 > F_3$	F_2
25	0.6791	0.7424	0.6461	0.6476	0.6682	$F_2 > F_1 > F_5 > F_4 > F_3$	F_2
50	0.6833	0.7437	0.6528	0.6551	0.6720	$F_2 > F_1 > F_5 > F_4 > F_3$	F_2

Table 3: Effect of the parameter r on decision making result using PFFWG operator

rived at the decision that all of the arithmetic and geometric aggregation operators for PFNs are contained in PF Frank aggregation operators, concerning the different values of r.

If we modify the value of the parameter r in the problem, we get different ranking results for the alternatives. For example, if we modify the value of the parameter r from 2 to 50, then using PFFWG operator we get the score values of the alternatives as $\Delta(\sigma_1) = 0.6833$, $\Delta(\sigma_2) = 0.7437$, $\Delta(\sigma_3) = 0.6528$, $\Delta(\sigma_4) = 0.6551$ and $\Delta(\sigma_5) = 0.6720$. Obviously, it can be obtained that the ranking position of the alternative F_4 changed from a bad position to a good position. But the PFWA and the PFWG operators are independent of the parameter r. So, the ranking order obtained with the help of those operators remains the same.

Based on the above comparison analysis, the approach in the present study is proved to be more flexible, compatible, and reliable than other existing procedures to control PF environment based MADM problems.

8 Conclusions

In this paper, we have studied MADM problems using PF information. We have developed Frank operations for PFSs and proposed a series of new aggregation operators, like, PFFWA operator, PFFOWA operator, PFFHA operator, PFFWG operator, PFFOWG operator, and PFFHG operator. Then, we have proposed an algorithm to deal with the MADM problem under the PF environment by using the PFFWA operator and the PFFWG operator. Finally, we have compared our proposed method with the existing approaches to exhibit its benefits and applicability.

In further research, we can study some new extensions of PFS, such as complex PFS, rough PFS. We can also extend them to other decision-making methods, such as COPRAS method [49], TOPSIS method [15], VIKOR [47] method, and so on, and apply them to deal with some real-life decision-making problems. We shall continue to investigate PF aggregation operators with the help of various t-norms and t-conorms.

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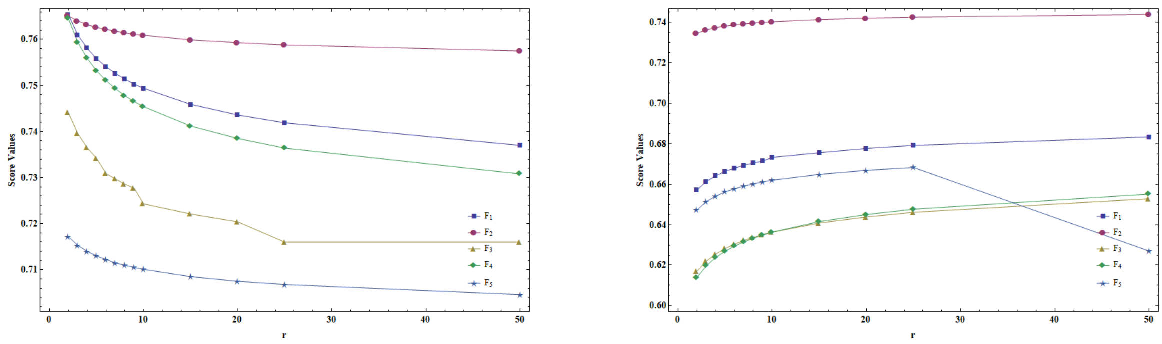

(a) Score of alternatives when $r \in [2, 50]$ based on PF-FWA operator

(b) Score of alternatives when $r \in [2, 50]$ based on PF-FWG operator

Figure 1: Pictorial representation of the ranking of the alternatives with different values of r

Aggregation Operators	$\Delta(\sigma_1)$	$\Delta(\sigma_2)$	$\Delta(\sigma_3)$	$\Delta(\sigma_4)$	$\Delta(\sigma_5)$	Ranking order	Optimal alternative
PFWA [37]	0.7731	0.7671	0.7522	0.7737	0.7206	$F_4 > F_1 > F_2 > F_3 > F_5$	F_4
PFWG [37]	0.6494	0.7312	0.6085	0.6040	0.6394	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2
Proposed method	0.6571	0.7345	0.6170	0.6139	0.6472	$F_2 > F_1 > F_5 > F_3 > F_4$	F_2

Table 4: Comparison table

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